Joining CNF-Formulas And Base-Hypergraphs

Stefan Porschen

Abstract—**The non-commutative joining operation for formulas and (base) hypergraphs recently introduced is considered here in a more general setting. Criteria for the (non-)diagonality of the joined hypergraph, and several results regarding the dependence of the orbit parameters from the joining-bijection are proven. Further, a commutative variant of this operation, called symmetric join, is defined. The properties of (dense) maximal non-diagonality, respectively minimal diagonality are characterized, for the special case that trivial hypergraphs are joined symmetrically.**

Index Terms—**hypergraph, CNF-satisfiability, transversal, orbit**

I. Introduction

The propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas is the basic NP-complete problem [6]. Via reduction numerous computational problems can be encoded as equivalent instances of CNF-SAT [7]. Besides the detection and the study of subclasses for which SAT can be decided efficiently such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested, co-nested formulas, and exact linear formulas etc. [2], [4], [5], [8], [9], [10], [11], [17], [18], one is interested in the structural properties of CNF-SAT.

In this paper the investigation of the structure of the class of base hypergraphs and its (fibre-)transversals is continued. A base hypergraph (BHG) underlies several CNF-formulas simultaneously, especially all its transversals. A diagonal BHG admits unsatisfiable transversals, whereas a non-diagonal one has satisfiable transversals, only. A structural parameter that is proposed to distinguish between diagonal BHGs is the number of orbits in the space of unsatisfiable transversals with respect to the action of the complementation group on CNF. In that manner, a hierarchy of diagonal BHGs appears, on whose ith level reside all instances admitting i orbits of unsatisfiable transversals [14]. Especially for $i = 0$, one has the class of non-diagonal BHGs, wherein the maximal non-diagonal instances reside, which even might be dense, representing the most extreme non-diagonal BHGs.

The present paper can be considered as a sequel to our paper [16], where in particular a (non-commutative) joining operation for CNFs, respectively BHGs, is introduced which mainly is applied there for the construction of specific classes of (dense) maximal non-diagonal, as well as of minimal diagonal BHGs.

Here we focus on the structure of that joining operation itself which is based on an intrinsic bijection. First several criteria for the (non)-diagonality of a joined BHG are provided some of which are proven to be independent of the bijection. We also address the question whether the orbit parameters are independent of the bijection. Specific instances are constructed establishing that the join of non-diagonal BHGs may have a non-diagonal, or a diagonal result depending on the bijection used. The logarithm of the number of all orbits is shown to be independent of the bijection if one joining component is uniform. However, if one joining component is diagonal, the number of orbits of unsatisfiabe transversals of the result is shown to depend on the bijection used, at least in the non-uniform case.

Further, a symmetric variant of the joining operation is discussed that is commutative and has a version for CNFs, respectively BHGs. We present and prove the main properties of symmetrically joined CNFs as well as their base hypergraphs. In this context, we also reveal the connection to the not-all-equal satisfiability problem for formulas resp. to the bicolorability problem (BIC) for hypergraphs. Here we show that the membership to BIC of a (symmetrically) joined BHG is independent of the bijection used for this join where both joining components have to be members of BIC.

Next, for the specific case of symmetrically joining trivial BHGs, necessary and sufficient criteria are shown for obtaining a non-diagonal, a minimal diagonal, or a (dense) maximal non-diagonal result. Also a local criterion for the non-diagonality of the symmetric join of not necessarily trivial BHGs is provided.

Several remarks with examples throughout illustrate the concepts. Finally, presenting some conclusions also offers directions for further investigations.

II. Notation and Preliminaries

For convenience, the basic terminology and notation are collected first: A (propositional) variable, x over $\{0, 1\}$ can appear as a positive literal x or as a negative literal $\overline{x}$, also called the complemented variable. A clause c is a finite non-empty disjunction of literals over mutually distinct variables; it is represented as a set $c = \{l_1, \ldots, l_k\}$, or simplifying, as a sequence of literals: $c = l_1 \cdots l_k$. Given x, $l(x)$ denotes the literal over x. A clause is positive (resp. negative) monotone, if all its literals are positive (resp. negative); monotone means any of these cases. A (conjunctive normal form) formula C is a finite conjunction of different clauses which is represented as a set of these clauses $C = \{c_1, \ldots, c_m\}$. Let CNF be the collection of all formulas. If all clauses of C are positive (negative) monotone, C is called a positive (negative) monotone formula, and called monotone if all its clauses are monotone. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). The size of C is $|C|$, and $\|C\| = \sum_{c \in C} |c|$ is its length.

For $U \subset V(C)$, the subformula $C(U) \subseteq C$ consists of all clauses possessing a literal over a variable in U. We identify $C(\{x\})$ with $C(x)$ whenever $x \in V(C)$. Restricting every clause $c \in C(U)$ to the literals over U only, denoted as $c[U]$, yields the (U-)retraction $C[U]$ of C [12]. Observe that the satisfiability of $C[U]$ implies that of $C(U)$.

Manuscript received February 8, 2025; revised March 24, 2025.

Stefan Porschen is a member of the Mathematics Group, at Department 4, HTW Berlin, D-10313 Berlin, Germany (e-mail: porschen@htw-berlin.de).

Given $C \in \text{CNF}$, SAT means to decide whether there is a (truth value) assignment $w : V(C) \to \{0,1\}$ s.t. there is no $c \in C$ all literals of which are set to 0, called a model of C; $\mathcal{M}(C)$ denotes the set of its models. $\text{SAT} \subseteq \text{CNF}$ is the collection of all formulas, for which there is a model, and $\text{UNSAT} := \text{CNF} \setminus \text{SAT}$. In particular $\mathcal{I} \subseteq \text{UNSAT}$ contains exactly all minimal unsatisfiable formulas [1]. Given a (here always finite) set V of propositional variables, an assignment w can be regarded as the clause $\{x^{w(x)} : x \in V\}$ of length $|V|$, where $x^0 := \bar{x}$, $x^1 := x$. Similarly, for $b \subseteq V$, we identify the restriction $w|b =: w(b)$ with the clause $\{x^{w(x)} : x \in b\}$. The set of all clauses of size $|V|$ is denoted as W_V which can also be regarded as the set of all mappings $V \to \{0,1\}$. The clause c^γ is obtained from c by complementing all its literals, in particular w^γ corresponds to $w \in W_V$, and $C^\gamma := \{c^\gamma : c \in C\}$; let $\mathcal{S} := \{C \in \text{CNF} : C = C^\gamma\}$.

It is $\mathcal{H}(C) = (V(C), B(C))$ the base hypergraph (BHG) of C, where $B(C) = \{V(c) : c \in C\}$ [12]. Let $C_b = \{c \in C : V(c) = b\}$ denote the fibre of C over b, thus $C = \bigcup_{b \in B(C)} C_b$. Given $\mathcal{H} = (V, B)$, its size is $|\mathcal{H}| := |B|$, and $\|\mathcal{H}\| := \sum_{b \in B} |b|$ is its length; $\mathcal{H}$ is (r-)regular, if every vertex is contained in exactly r (hyper-)edges. If every edge has fixed size k, then $\mathcal{H}$ is (k-)uniform. Let $U \subset V$ s.t. the non-empty $b \cap U$, for $b \in B$, are mutually distinct. Then $\mathcal{H}[U] := (U, B[U])$ is the (U-)retraction of $\mathcal{H}$ where $B[U] := \{b \cap U : b \in B, b \cap U \neq \emptyset\}$.

Let $\mathfrak{H}$ be the collection of all BHGs, and let $\mathfrak{H}^c$ be the subclass of all connected instances. Edges of the bipartite incidence graph $I_\mathcal{H}$ of $\mathcal{H}$ are denoted as $v - b$, for $v \in V$, $b \in B$ with $v \in b$. Given a matching M of $I_\mathcal{H}$, we denote the set of all its V-members, B-members as $M(V)$, $M(B)$ respectively.

A BHG is linear if distinct edges pairwise intersect in at most one vertex; if one is the size of all these intersections it even is exact linear. A hypergraph is loopless [3] if none of its edges has size one. Let $\mathfrak{H}^\ell$ denote the class of all BHGs, whose instances consist of loops only. When adding a loop to $\mathcal{H}$, the notation is simplified by writing $\mathcal{H} \cup \{x\}$ instead of $\mathcal{H} \cup \{\{x\}\}$.

A formula C s.t. $|C_b| = 1$, for all $b \in B(C)$, is (exact) linear if $\mathcal{H}(C)$ is (exact) linear [17], recall that a linear formula cannot contain complementary unit clauses.

A well-known variant of SAT, namely the problem not-all-equal SAT (NAESAT), asks whether $C \in \text{CNF}$ admits a model s.t. each clause has a literal set to 0. Restricted to monotone formulas, which can be identified with their BHGs, NAESAT coincides with the bicolorability problem (BIC) for hypergraphs: $\mathcal{H} \in \text{BIC}$ iff there is a proper 2-coloring of $V(\mathcal{H})$, s.t. no edge is monochromatic.

As usual $K_\mathcal{H} := \bigcup_{b \in B} W_b$ is the set of all clauses over $\mathcal{H} = (V, B)$. A (fibre-)transversal over $\mathcal{H}$, is a formula $F \subset K_\mathcal{H}$ s.t. $|F \cap W_b| = 1$, for each $b \in B$. Hence F contains exactly one clause of each fibre W_b of $K_\mathcal{H}$. Let this clause, say c, be refered to as F_b thereby identifying the fibre $F_b = \{c\}$ with the clause c itself. Note that in general $F(b) \neq F_b$. The set of all transversals over $\mathcal{H}$ is denoted as $\mathcal{F}(\mathcal{H})$. A transversal F is compatible if $\bigcup_{b \in B} F_b \in W_V$ whose collection is $\mathcal{F}_{\text{comp}}(\mathcal{H}) \subseteq \text{SAT}$. A transversal F is diagonal if $F \cap F' \neq \emptyset$, for all $F' \in \mathcal{F}_{\text{comp}}(\mathcal{H})$; $\mathcal{F}_{\text{diag}}(\mathcal{H})$ is the set of all diagonal transversals of $\mathcal{H}$. In $\mathcal{F}_{\text{diag}}(\mathcal{H})$ exactly all the unsatisfiable transversals over $\mathcal{H}$ are contained. So, $\mathcal{H}$ is called diagonal if $\mathcal{F}_{\text{diag}}(\mathcal{H}) \neq \emptyset$, and in this case it is minimal diagonal if none of its sub-hypergraphs is diagonal. Let $\mathfrak{H}_{\text{diag}}$ be the class of all diagonal BHGs, and $\mathfrak{H}_{\text{mdiag}}$ denote the subclass of all minimal diagonal instances. For a fixed vertex set V, let $\mathcal{K}_V := (V, 2^V \setminus \{\emptyset\})$ denote the complete BHG. Let $\mathcal{K}_n$ represent any instance of $\mathcal{K}_V$, for $|V| = n$. It is $\mathcal{K}_n \in \mathfrak{H}_{\text{diag}} \setminus \mathfrak{H}_{\text{mdiag}}$, if $n > 2$, and only $\mathcal{K}_2 \in \mathfrak{H}_{\text{mdiag}}$. A non-diagonal BHG $\mathcal{H} = (V, B)$ is maximal non-diagonal, if there is any new $b \subseteq V$ s.t. $\mathcal{H} \cup \{b\}$ becomes diagonal. If the same is true for every new $b \subseteq V$ it even is dense maximal non-diagonal w.r.t. $\mathcal{K}_V$ [16]. We use $\mathfrak{H}_{\text{maxnd}}$ for the class of all maximal non-diagonal BHGs.

The group of variable complementation G_V induces an action on the set of all formulas with variable set V [15]. A (G_V-)orbit w.r.t. this action is denoted as $\mathcal{O}$. Given $F \in \mathcal{F}(\mathcal{H})$, for $\mathcal{H} = (V, B)$, its orbit is denoted as $\mathcal{O}(F)$. The number of all orbits in $\mathcal{F}_{\text{diag}}(\mathcal{H})$ is defined as $\delta(\mathcal{H})$ [13]. One has $\delta = 0$ for all non-diagonal instances, collected in $\mathfrak{H}_0$. More generally, $\mathfrak{H}_i$ collects all BHGs with $\delta = i$; for $\delta = 1$ the instances are called simple and collected in $\mathfrak{H}_{\text{simp}}$. Further orbit parameters of $\mathcal{H}$ are $\beta(\mathcal{H}) = \|\mathcal{H}\| - |V|$, $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$, as well as $\rho(\mathcal{H})$ [14]. It is $\omega(\mathcal{H}) = 1 + \delta(\mathcal{H}) + \rho(\mathcal{H})$.

We use $[n] := \{1, \ldots, n\}$, $[n]_0 := [n] \cup \{0\}$, where n is a positive integer, and $\mathbb{N}$ for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For finite sets A, B of equal cardinality, let $\text{Bij}(A, B)$ denote the collection of all bijections $A \to B$, in particular $S_n = \text{Bij}([n], [n])$ means the symmetric group of degree n.

For convenience, FPP(1) symbolizes a BHG which is a simple triangle, hence in particular it is 2-regular, 2-uniform and exact linear. (It could be regarded as isomorphic to a "finite projective plane of order 1", which does not meet all axioms of projective (plane) geometry, cf. [19], [17], [16].)

III. A Mapping for Joining CNFs

The joining operation as introduced in [16] here is considered in a slightly more abstract setting. For fixed $m \in \mathbb{N}$, and non-empty variable set V, let each member of

$$\text{CNF}(V, m) := \{C \in \text{CNF} : V(C) = V, |C| = m\}$$

be labeled over $[m]$. Let V_i, $i \in [2]$, denote disjoint non-empty variable sets. Define

$$\bigotimes_m^{V_1, V_2} : \text{CNF}(V_1, m) \times \text{CNF}(V_2, m) \times S_m \to \text{CNF}$$

via

$$\bigotimes_m^{V_1, V_2}(C_1, C_2, \sigma) := \bigcup_{j \in [m]} c_j \otimes \hat{c}_{\sigma(j)}$$

which is called the (σ-)join of $C_1 := \{c_j : j \in [m]\} \in \text{CNF}(V_1, m)$ and $C_2 := \{\hat{c}_j : j \in [m]\} \in \text{CNF}(V_2, m)$, for $\sigma \in S_m$, where

$$c_j \otimes \hat{c}_{\sigma(j)} := \left\{c_j \cup \{l\} : l \in \hat{c}_{\sigma(j)}\right\} \cup \left\{\hat{c}^\gamma_{\sigma(j)}\right\}$$

For fixed m, V_1, V_2, set

$$\bigotimes_m^{V_1, V_2}(C_1, C_2, \sigma) =: C_1 \otimes_\sigma C_2$$

(cf. [16]). If $m = 1$, we simply write $C_1 \otimes C_2$. Clearly, σ induces a unique member $\tilde{\sigma} \in \text{Bij}(C_1, C_2)$, and vice versa. Evidently, the labeled edge set of a BHG is a labeled (positive) monotone formula. So, setting $\mathfrak{H}(V,m) := \{\mathcal{H} = (V,B) \in \mathfrak{H} : |B| = m\}$, it is

$$\bigotimes_{m}^{\mathfrak{H}\ V_1,V_2} : \mathfrak{H}(V_1,m) \times \mathfrak{H}(V_2,m) \times S_m \to \mathfrak{H}$$

$$\bigotimes_{m}^{\mathfrak{H}\ V_1,V_2}(\mathcal{H}_1,\mathcal{H}_2,\tau) := \left(V_1 \cup V_2, B\left(\bigotimes_{m}^{V_1,V_2}(B_1,B_2,\tau)\right)\right)$$

where $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(V_i, m)$, $i \in [2]$, $\tau \in S_m$. For fixed m, V_1, V_2, that is abbreviated as

$$\mathcal{H}_1 \otimes_\tau \mathcal{H}_2 := (V_1 \cup V_2, B(B(\mathcal{H}_1) \otimes_\tau B(\mathcal{H}_2)))$$

As above, τ induces a unique member $\tilde{\tau} \in \mathrm{Bij}(\mathcal{H}_1, \mathcal{H}_2)$, and vice versa.

The example in Remark 5 in [16] shows that already for $m = 2$, $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}_0$ with $\beta(\mathcal{H}_2) > 0$ and even $\beta(\mathcal{H}_1) = 0$, there is $\tau \in S_m$ s.t. $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2 \in \mathfrak{H}_{\mathrm{diag}}$. According to Prop. 13 (3) in [16], then one even has $F_1 \otimes_\tau F_2 \in \mathrm{SAT}$, for all $F_i \in \mathcal{F}(\mathcal{H}_i)$, $i \in [2]$.

The next result lists several conditions for the (non-)diagonality of a joined BHG.

Theorem 1: Let $m \in \mathbb{N}$, $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(V_i, m)$, $i \in [2]$, $V_1 \cap V_2 = \emptyset$ and $\mathcal{H}_\tau := \mathcal{H}_1 \otimes_\tau \mathcal{H}_2$, $\tau \in S_m$, then:

1. *$\mathcal{H}_\tau \in \mathfrak{H}_0$ implies $\mathcal{H}_i \in \mathfrak{H}_0$, $i \in [2]$.*
2. *If $\mathcal{H}_1 \in \mathfrak{H}_0$ and $\beta(\mathcal{H}_2) = 0$ then $\mathcal{H}_\tau \in \mathfrak{H}_0$.*
3. *If $\beta(\mathcal{H}_1) = 0$ s.t. $|\tilde{\tau}(b)| < 2^{|b|}$, for all $b \in B_1$, and $\mathcal{H}_2 \in \mathfrak{H}_0$ then $\mathcal{H}_\tau \in \mathfrak{H}_0$.*
4. *If $\beta(\mathcal{H}_1) = 0$ and $\mathcal{H}_2 \in \mathfrak{H}_0$ s.t.*
$$\max\left\{|\hat{b}| : \hat{b} \in B_2\right\} < \min\left\{2^{|b|} : b \in B_1\right\}$$
then $\mathcal{H}_\tau \in \mathfrak{H}_0$.
5. *If $F[V_1] \in \mathrm{SAT}$, for all $F \in \mathcal{F}(\mathcal{H}_\tau)$, and $\mathcal{H}_2 \in \mathfrak{H}_0$ then $\mathcal{H}_\tau \in \mathfrak{H}_0$.*
6. *If $\mathcal{H}_\tau \in \mathfrak{H}_0$, and there is $F_2 \in \mathcal{F}(\mathcal{H}_2)$ s.t. $|\mathcal{M}(F_2)| = 1$, then $F[V_1] \in \mathrm{SAT}$, for all $F \in \mathcal{F}(\mathcal{H}_\tau)$.*
7. *Let $\mathcal{H}_i \in \mathfrak{H}_0$, $i \in [2]$, s.t. there is $F_2 \in \mathcal{F}(\mathcal{H}_2)$ with $|\mathcal{M}(F_2)| = 1$. Then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$ iff there exists $F \in \mathcal{F}(\mathcal{H}_\tau)$ s.t. $F[V_1] \in \mathrm{UNSAT}$.*
8. *Let $\mathcal{H}_i \in \mathfrak{H}_0$, $i \in [2]$, s.t. there is $b \in B_1$ with $|\tilde{\tau}(b)| \geq 2^{|b|}$, and $F_2 \in \mathcal{F}(\mathcal{H}_2)$ with $|\mathcal{M}(F_2)| = 1$, then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$.*
9. *Let $\beta(\mathcal{H}_1) = 0$, $\mathcal{H}_2 \in \mathfrak{H}_0$ s.t. there is $F_2 \in \mathcal{F}(\mathcal{H}_2)$ with $|\mathcal{M}(F_2)| = 1$. Then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$ iff there exists $b \in B_1$ s.t. $|\tilde{\tau}(b)| \geq 2^{|b|}$.*

PROOF. (1) directly is implied by Prop. 13 (3) in [16].

By the definition of $\mathcal{H}_\tau$, given $F \in \mathcal{F}(\mathcal{H}_\tau)$ there is a unique $F_2 \in \mathcal{F}(\mathcal{H}_2)$ s.t. $F_2 \subsetneq F$, namely the restriction of F to B_2. Thus, we always have the partition $F = F(V_1) \cup F_2$. On that basis addressing (2), assume $\mathcal{H}_1 \in \mathfrak{H}_0$, $\beta(\mathcal{H}_2) = 0$, and let $F \in \mathcal{F}(\mathcal{H}_\tau)$ be chosen arbitrarily. So $\mathcal{H}_2 \in \mathfrak{H}_0$, and $F = \bigcup_{\hat{b} \in B_2} F(\hat{b})$. Let $\hat{c}_{\hat{b}}$ denote the unique clause of F_2 in $F(\hat{b})$. Moreover consider the retraction $C := F[V_1] \subset \bigcup_{b \in B_1} W_b$. Depending on the size of $\tilde{\tau}(b)$, to each clause $c \in C_b$ there can correspond several clauses in $F(\tilde{\tau}(b))$, each of which arises from c by adding exactly one literal over a variable in $\tilde{\tau}(b)$, s.t. these variables are mutually distinct. The collection of these clauses is denoted as $F^c(\tilde{\tau}(b))$.

Case (i): Assume that $|C_b| \geq 2$, for all $b \in B_1$. Let T be a compatible transversal over $\mathcal{H}_1$. According to Thm. 1 and its proof in [12], setting all literals of T to 0 leaves unsatisfied at most one clause in every fibre, say $c_b \in C_b$, for all $b \in B_1$. Then all literals over V_2 in the clauses of $F^{c_b}(\tilde{\tau}(b))$ have to be assigned to 1, for satisfying all clauses of $F(\tilde{\tau}(b))$. These assignments can be performed independently, because $\tilde{\tau}(b)$, $b \in B_1$, are mutually disjoint. Hence, $F \setminus F_2 = F(V_1)$ is solved so far. Further, there exists a clause $c'_b \neq c_b$ in C_b among the satisfied ones, for all $b \in B_1$. So the unique variable over V_2 of an arbitrary clause in $F^{c'_b}(\tilde{\tau}(b))$ can be assigned independently to solve $\hat{c}_{\tilde{\tau}(b)} \in F_2$, if still necessary. Again this is possible because all clauses in F_2 are variable-disjoint; thus $F \in \mathrm{SAT}$.

Case (ii): Assume that

$$B_1^* := \{b \in B_1 : |C_b| = 1\} \neq \emptyset$$

which in particular is the case if $\mathcal{H}_2$ has loops. Let $B'_1 := B_1 \setminus B_1^*$. If $B'_1 = \emptyset$, we are done because then $C = F[V_1] \in \mathcal{F}(\mathcal{H}_1)$ is satisfiable, so the subformula $F(V_1) \in \mathrm{SAT}$ can be solved over V_1 only, and F_2 can be solved over V_2 independently. Otherwise, we have $B_2^* := \tilde{\tau}(B_1^*) \neq \emptyset$ and $B'_2 := \tilde{\tau}(B'_1) \neq \emptyset$, as disjoint parts. So,

$$\mathcal{H}_\tau = (\mathcal{H}_1^* \otimes_\tau \mathcal{H}_2^*) \cup (\mathcal{H}'_1 \otimes_\tau \mathcal{H}'_2)$$

as disjoint union implying the corresponding decomposition $F = F^* \cup F'$, where F^*, F' is the restriction of F to $\mathcal{H}_1^* \otimes_\tau \mathcal{H}_2^*$, $\mathcal{H}'_1 \otimes_\tau \mathcal{H}'_2$, respectively. Let w be a (partial) model of F^* which always exists as previously stated. Moreover, if necessary w can be extended to all of V_1, s.t. a compatible transversal over $\mathcal{H}_1$ is determined, from which as shown in case (i) a model for all of F can be constructed, because the F_2-clauses are mutually disjoint.

For proving (3), let $F \in \mathcal{F}(\mathcal{H}_\tau)$ with $F = F(V_1) \cup F_2$, where $F_2 \in \mathcal{F}(\mathcal{H}_2)$. Assume $\beta(\mathcal{H}_1) = 0$, and $|\tilde{\tau}(b)| < 2^{|b|}$, for all $b \in B_1$. Then one has $F[b] \subsetneq W_b$ which is satisfiable independently for each $b \in B_1$. Therefore $F(b) \in \mathrm{SAT}$, for all $b \in B_1$, so $F(V_1) \in \mathrm{SAT}$ imlying $F \in \mathrm{SAT}$ as $\mathcal{H}_2 \in \mathfrak{H}_0$. Evidently (4) is implied by (3).

For $F \in \mathcal{F}(\mathcal{H}_\tau)$, with $F = F(V_1) \cup F_2$ one has $F[V_1] \in \mathrm{SAT}$, so $F(V_1) \in \mathrm{SAT}$, by the assumption of (5). Also $F_2 \in \mathrm{SAT}$, as $\mathcal{H}_2 \in \mathfrak{H}_0$, implying $F \in \mathrm{SAT}$, so (5) is true.

Every $F \in \mathcal{F}(\mathcal{H}_\tau)$ with $F = F(V_1) \cup F_2$, determines a further transversal $\tilde{F}_2 \in \mathcal{F}(\mathcal{H}_2)$, besides F_2: For each $b \in B_1$, let the set of literals over V_2 in all clauses of F over $B(b \otimes \tilde{\tau}(b)) \setminus \tilde{\tau}(b)$ be regarded as the clause of $\tilde{F}_2$ over $\tilde{\tau}(b)$. Now let the assumption of (6) be valid, but suppose there is $F \in \mathcal{F}(\mathcal{H}_\tau)$ s.t. $F[V_1] \in \mathrm{UNSAT}$. Let $F'_2 \in \mathcal{F}(\mathcal{H}_2)$ have the unique model w_2. Let $F' \in \mathcal{F}(\mathcal{H}_\tau)$ be obtained from F by substituting F_2 by F'_2 and $\tilde{F}_2$ by $w_2^\gamma(B_2)$. Since $F'[V_1] = F[V_1] \in \mathrm{UNSAT}$, $F'(V_1)$ can be satisfied only via the literals over V_2. But as F'_2 can be satisfied only by w_2, which assigns all literals in $\tilde{F}'_2 = w_2^\gamma(B_2)$ to 0, it is $F' \in \mathrm{UNSAT}$. Thus $\mathcal{H}_\tau$ is diagonal providing a contradiction, establishing (6).

Combining (5) and (6) one obtains (7). Regarding (8), assume that there is $b \in B_1$ s.t. $|\tilde{\tau}(b)| \geq 2^{|b|}$. So there is $F \in \mathcal{F}(\mathcal{H}_\tau)$ with $F[b] \supset W_b \in \mathrm{UNSAT}$ implying $F[b] \subseteq F[V_1] \in \mathrm{UNSAT}$ and yielding (8) relying on (7). Finally, combining (3) and (8) yields (9). □

Remark 1: 1. The converse of assertion (2) in general is false: For $m = 2$, let

$$B_1 := \{x_1x_2, x_2x_3\}, \quad B_2 = \{y_1y_2, y_2y_3\}$$

then for $\mathcal{H}_\tau = \mathcal{H}_1 \otimes_\tau \mathcal{H}_2$ with $\tau = \mathrm{id} \in S_2$ it is

$$B(\mathcal{H}_{\mathrm{id}}) = \{x_1x_2y_1, x_1x_2y_2, y_1y_2, x_2x_3y_2, x_2x_3y_3, y_2y_3\}$$

yielding $\mathcal{H}_{\mathrm{id}} \in \mathfrak{H}_0$. Indeed, the 6 clauses of any of its transversals in the above order can be solved independently via x_1, x_2, y_1, y_2, x_3, y_3, respectively. The same is valid for the second permutation in S_2; further it is $\beta(\mathcal{H}_i) > 0$, $i \in [2]$. (More generally, one has Thm. 2 (1), (4), below.)

2. The assertion (5) is weaker than (3).

3. The example stated above in 1. also demonstrates that the converse of assertion (5) in general is invalid: Let F be a transversal over $\mathcal{H}_{\mathrm{id}}$ s.t.

$$F[V_1] = \{x_1\bar{x}_2, \bar{x}_1\bar{x}_2, x_2x_3, x_2\bar{x}_3\}$$

being a member of UNSAT. *However, one could regard (6) as a weak reverse of (5).*

4. The second assumption in assertions (6), (7) especially implies that $\mathcal{H}_2$ is dense maximal non-diagonal, if $V_2| \geq 2$, according to Theorem 5 in [16].

Theorem 2: Let $m \geq 2$, $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(V_i, m)$, $i \in [2]$, $V_1 \cap V_2 = \emptyset$, $\tau \in S_m$, and $\mathcal{H}_\tau$ as above.

(1) *For $\mathcal{H}_1 \in \mathfrak{H}_0^{\mathrm{c}}$ s.t. every $b \in B_1$ has two unique vertices, and integer $\hat{m} \geq 3m$, there exists $\mathcal{H}_2 \in \mathfrak{H}_0^{\mathrm{c}}$ with $\mathcal{H}_\tau \in \mathfrak{H}_0$, and $|\mathcal{H}_\tau| \geq \hat{m}$, for all $\tau \in S_m$.*
(2) *For each $\mathcal{H}_1 \in \mathfrak{H}_0$, $\beta(\mathcal{H}_1) = 0$ having 2 loops, there exists $\mathcal{H}_2 \in \mathfrak{H}_0$ s.t. $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$, for all $\tau \in S_m$.*
(3) *Let $m \geq 4$. For each $\mathcal{H}_1 \in \mathfrak{H}_0$, $\beta(\mathcal{H}_1) = 0$ admitting $b \in B_1$ with $|b| \leq \lfloor \log_2(m-1) \rfloor$, there exist $\mathcal{H}_2 \in \mathfrak{H}_0$, $\tau \in S_m$ s.t. $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$.*
(4) *For each $\mathcal{H}_2 \in \mathfrak{H}_0^{\mathrm{c}}$, there exists $\mathcal{H}_1 \in \mathfrak{H}_0$, s.t. $\mathcal{H}_\tau \in \mathfrak{H}_0$, for all $\tau \in S_m$.*

PROOF. Regarding (1), it suffices to provide an appropriate large, connected $\mathcal{H}_2$ s.t. the incidence graph $I(\mathcal{H}_\tau)$ of $\mathcal{H}_\tau =: (V_\tau, B_\tau)$ admits a perfect matching M, for each $\tau \in S_m$: Given $F \in \mathcal{F}(\mathcal{H}_\tau)$, then each variable in $M(V_\tau)$ uniquely solves the clause of F over the corresponding hyperedge in $M(B_\tau)$, respectively. To that end, let $\mathcal{H}_2$ be k-uniform, with $k \geq \lceil \frac{\hat{m}}{m} - 1 \rceil \geq 2$. So

$$|\mathcal{H}_\tau| = \|B_2\| + m = (k+1)m \geq \hat{m}$$

where Thm. 15 (2) in [16] has been used. Further, the connectedness of $\mathcal{H}_2$, and the existence of $M = M(k)$ as required can be assured as follows: For $k = 2$, set

$$B_2 := \left\{\hat{b}_j := y_j y_{j+1} : j \in [m]\right\}$$

which is a simple path of $m \geq 2$ edges. Thus for an arbitrary $\tau \in S_m$, one has $B_\tau = \bigcup_{j \in [m]} B(b_j \otimes \hat{b}_{\tau(j)})$, where

$$B(b_j \otimes \hat{b}_{\tau(j)}) = \left\{b_j \cup \{y_{\tau(j)}\}, b_j \cup \{y_{\tau(j+1)}\}, \hat{b}_{\tau(j)}\right\}$$

$j \in [m]$. Let $x(b_j), x'(b_j)$ denote the unique vertices in $b_j \in B_1$, $j \in [m]$. Thus

$$\begin{aligned} M(2) &= \bigcup_{j \in [m]} \Big\{ x(b_j) - (b_j \cup \{y_{\tau(j)}\}), \\ &\qquad x'(b_j) - (b_j \cup \{y_{\tau(j+1)}\}), y_{\tau(j)} - \hat{b}_{\tau(j)} \Big\} \end{aligned}$$

For $k \geq 3$, enlarge $\hat{b}_j$ by the unique vertices $\{z_l^j : l \in [k-2]\}$, for every $j \in [m]$. Hence the adapted B_2 still is a path, now of hyperedges. Then it is

$$\begin{aligned} B(b_j \otimes \hat{b}_{\tau(j)}) &= \left\{b_j \cup \{y_{\tau(j)}\}, b_j \cup \{y_{\tau(j+1)}\}, \hat{b}_{\tau(j)}\right\} \\ &\cup \left\{b_j \cup \{z_l^{\tau(j)}\} : l \in [k-2]\right\} \end{aligned}$$

$j \in [m]$. Setting

$$M(k) = M(2) \cup \bigcup_{j \in [m]} \left\{z_l^{\tau(j)} - (b_j \cup \{z_l^{\tau(j)}\}) : l \in [k-2]\right\}$$

provides a perfect matching as required, yielding (1).

For every integer $m \geq 2$, according to Thm. 8 (1) in [16] and its proof, there exists $\mathcal{H}_2$ of size m, $\beta(\mathcal{H}_2) \neq 0$:

$$V_2 = \{x_j : j \in [m]\}, \quad B_2 = \{x_1\} \cup \{x_j x_{j+1} : j \in [m-1]\}$$

admitting a transversal that has exactly one model, in which all variables are set to 1. Thus every edge of $\mathcal{H}_2$ has size at most 2. One obtains (2) relying on Thm. 1 (8), as only one of the loops in $\mathcal{H}_1$ can be mapped to the unique loop in $\mathcal{H}_2$, by $\tau \in S_m$.

Concerning (3), we modify the previous BHG by substituting m by $m-1 \geq 3$, and adding one edge containing all $m-1$ vertices ensuring the size m: $V_2 := \{x_j : j \in [m-1]\}$,

$$B_2 := \{x_1\} \cup \{x_j x_{j+1} : j \in [m-2]\} \cup \left\{\hat{b} := x_1 \ldots x_{m-1}\right\}$$

If the transversal, as mentioned above, is enlarged e.g. by the monotone clause $\hat{b}$, the unique model is maintained. Choose $\tau \in S_m$ s.t. $\tilde{\tau}(b) = \hat{b}$ then

$$2^{|b|} \leq m - 1 = |\tilde{\tau}(b)|$$

and the assertion follows via Thm. 1 (8).

Finally regarding (4), it suffices to choose $\mathcal{H}_1 \in \mathfrak{H}_0$ with $\beta(\mathcal{H}_1) = 0$ s.t., for all $b \in B_1$, it is

$$|b| > \max\left\{\lceil \log_2 |\hat{b}| \rceil : \hat{b} \in B_2\right\}$$

according to Thm. 1 (4). □

Next, a first characterization of minimal diagonality is provided on behalf of Thm. 11 in [16]:

Proposition 1: Let m, $\mathcal{H}_i$, $i \in [2]$, τ, $\mathcal{H}_\tau$ as in Thm. 1 and $\beta(\mathcal{H}_2) = 0$. Then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{mdiag}}$ iff $\mathcal{H}_1 \in \mathfrak{H}_{\mathrm{mdiag}}$.

PROOF. In Thm. 11 in [16] the same assertion is stated using the further assumption that $\mathcal{H}_1$ is diagonal, which is dropped here. So, the right-to-left implication directly follows from the cited Thm. 11. The reverse one follows by contraposition: If $\mathcal{H}_1 \notin \mathfrak{H}_{\mathrm{mdiag}}$, it either is in $\mathfrak{H}_0$ so $\mathcal{H}_\tau \in \mathfrak{H}_0$ using Thm. 1 (2). Or $\mathcal{H}_1 \in \mathfrak{H}_{\mathrm{diag}} \setminus \mathfrak{H}_{\mathrm{mdiag}}$, then again we are done by the cited Thm. 11. □

Concerning a characterization regarding the maximal non-diagonality, a first fact is provided next:

Proposition 2: Let m, $\mathcal{H}_i$, $i \in [2]$, τ, $\mathcal{H}_\tau$ as in Thm. 1. If $\mathcal{H}_1 \in \mathfrak{H}_0$ has a loop and $\beta(\mathcal{H}_2) = 0$, then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{maxnd}}$.

PROOF. On behalf of Thm. 1 (2), it is $\mathcal{H}_\tau \in \mathfrak{H}_0$. Let $b \in B_1$ be a loop, then $\mathcal{H}(b \otimes \tilde{\tau}(b)) \subseteq \mathcal{H}_\tau$, but $b \notin B(\mathcal{H}_\tau)$. Obviously $\mathcal{H}_\tau \cup \{b\} \in \mathfrak{H}_{\mathrm{diag}}$. □

IV. FIBRE-RESPECTING FORMULA-JOINS

At least if C_1, C_2 are transversals equally labeled as their BHGs, one has

$$\mathcal{H}(C_1 \otimes_\sigma C_2) = \mathcal{H}(C_1) \otimes_\sigma \mathcal{H}(C_2)$$

cf. La. 11 in [16]. In general, this equality fails or its right hand side even might not exist. E.g., let $C_1 = W_{V_1}$ with $V_1 := \{x_1, x_2\}$ hence $\mathcal{H}_1 := \mathcal{H}(C_1) \in \mathfrak{H}(V_1, 1)$. If $C_2 \in \mathcal{F}(\mathcal{H}_2)$ with

$$B(\mathcal{H}_2) := \{u_1u_2u_3, v_1v_2v_3, y_1y_2y_3, z_1z_2z_3\}$$

then $\mathcal{H}_2 \in \mathfrak{H}(V_2, 4)$. So, $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}_0$ even are trivial BHGs, and $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2$ cannot be defined, for any τ. Whereas $C_1 \otimes_\sigma C_2$ is well-defined, and

$$\mathcal{H}(C_1 \otimes_\sigma C_2) \in \mathfrak{H}(V_1 \cup V_2, 16)$$

for every $\sigma \in S_4$. Moreover one easily verifies $C_1 \otimes_\sigma C_2 \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}(C_1 \otimes_\sigma C_2))$ by observing $C_1 \in \mathrm{UNSAT}$ and using Prop. 13 (3),(5) in [16].

Towards clarifying the situation in case that $C_1 \otimes_\sigma C_2$ and $\mathcal{H}(C_1) \otimes_\tau \mathcal{H}(C_2)$ are well defined, for appropriate σ, τ respectively, a useful notion is introduced:

Definition 1: For $m, m' \in \mathbb{N}$, let $C_i \in \mathrm{CNF}(V_i, m)$, s.t. $\mathcal{H}(C_i) \in \mathfrak{H}(V_i, m')$, $i \in [2]$, $V_1 \cap V_2 = \emptyset$. Given $\tau \in S_{m'}$, a σ-join $C_1 \otimes_\sigma C_2$, for $\sigma \in S_m$, is called fibre-respecting (w.r.t. τ) *if $V(\tilde{\sigma}(c)) = \tilde{\tau}(V(c))$, for all $c \in C_1$.*

Lemma 1: For m, m', C_i, $\mathcal{H}(C_i)$, V_i, $i \in [2]$, as in Def. 1 and $C_\sigma := C_1 \otimes_\sigma C_2$, $\sigma \in S_m$, one has:

- (i) *C_σ is fibre-respecting w.r.t. $\tau \in S_{m'}$, iff $\tilde{\sigma}(C_{1b}) = C_{2\tilde{\tau}(b)}$, for every $b \in B_1$. In this case, $|C_{1b}| = |C_{2\tilde{\tau}(b)}|$, for every $b \in B_1$.*
- (ii) *If $m' = 1$, i.e., C_i is a fibre-formula, $i \in [2]$, then $B(C_\sigma) = B(B(C_1) \otimes B(C_2))$, for every $\sigma \in S_m$, recalling the convention for $m' = 1$.*

PROOF. (i) directly follows from the definition.

Obviously $B(B(C_1) \otimes B(C_2)) \subseteq B(C_\sigma)$, for every $\sigma \in S_m$, and C_σ then is fibre-respecting. Let $B(C_i) = \{b_i\}$, $i \in [2]$. So if $b \in B(C_\sigma)$ then either

$$b = V(\tilde{\sigma}(c)) = b_2 \in B(C_2) \subseteq B(B(C_1) \otimes B(C_2))$$

or $b = V(c \cup l) = b_1 \cup \{x\}$, where $x = V(l) \in b_2$, meaning $b \in B(B(C_1) \otimes B(C_2))$. □

Proposition 3: Let m, m', C_i, $\mathcal{H}(C_i)$, V_i, $i \in [2]$, as in Def. 1 and $C := C_1 \otimes_\sigma C_2$, for $\sigma \in S_m$. Then

$$\mathcal{H}(C) = \mathcal{H}(C_1) \otimes_\tau \mathcal{H}(C_2)$$

iff C is fibre-respecting w.r.t. $\tau \in S_{m'}$. The latter in particular is valid if C_1, C_2 are transversals equally labeled as their BHGs, and $\tau = \sigma$.

PROOF. Clearly, for every $C \in \mathrm{CNF}$, it is $|\mathcal{H}(C)| \le |C|$ which is an equality only if C is a transversal. So the last assertion is implied by La. 11 in [16]. Regarding the first assertion, let $\mathcal{H}_i := \mathcal{H}(C_i) =: (V_i, B_i)$, $i \in [2]$, and assume that C is fibre-respecting w.r.t. $\tau \in S_{m'}$ then by La. 1 (i)

$$C = \bigcup_{b \in B_1} C_{1b} \otimes_{\sigma_b} C_{2\tilde{\tau}(b)}$$

where σ_b denotes the restriction of σ to C_{1b}. So,

$$\begin{aligned} B(C) &= \bigcup_{b \in B_1} B\left(C_{1b} \otimes_{\sigma_b} C_{2\tilde{\tau}(b)}\right) \\ &= \bigcup_{b \in B_1} B\left(B(C_{1b}) \otimes B(C_{2\tilde{\tau}(b)})\right) \\ &= B\left(B(C_1) \otimes_\tau B(C_2)\right) \end{aligned}$$

where La. 1 (ii) has been used.

Next, assume that the asserted equality is true, and suppose that C fails to be fibre-respecting. Hence there is $b \in B_1$ and $c \in C_{1b}$ with $\breve{b} := V(\tilde{\sigma}(c)) \neq \tilde{\tau}(b)$. So,

$$\left\{b \cup \{x\} : x \in \breve{b}\right\} \subset B(c \otimes \tilde{\sigma}(c)) \subseteq B(C)$$

but

$$\left\{b \cup \{x\} : x \in \breve{b}\right\} \not\subset B(B_1 \otimes_\tau B_2)$$

yielding a contradiction. □

V. S_m-Dependence of the Orbit Parameters

Focusing on $\{\mathcal{H}_1 \otimes_\tau \mathcal{H}_2 : \tau \in S_m\}$, primarily the question arises, which properties of the joined BHG depend on $\tau \in S_m$. A first result, regarding the logarithm of the number of all orbits is stated next.

Proposition 4: Let $m \in \mathbb{N}$, $\mathcal{H}_i \in \mathfrak{H}(V_i, m)$ with fixed labeled edge set, $i \in [2]$, $V_1 \cap V_2 = \emptyset$. Then $\beta(\mathcal{H}_\tau)$ is independent of $\tau \in S_m$ iff one of $\mathcal{H}_1, \mathcal{H}_2$ is uniform, where $\mathcal{H}_\tau := \mathcal{H}_1 \otimes_\tau \mathcal{H}_2$.

PROOF. Setting $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, it is

$$\beta(\mathcal{H}_\tau) = \|\mathcal{H}_\tau\| - (|V_1| + |V_2|)$$

Due to Prop. 13 (2) in [16], one has

$$\|\mathcal{H}_\tau\| = 2\|\mathcal{H}_2\| + \sum_{b \in B_1} |b| \cdot |\tilde{\tau}(b)|$$

Thus, if $\mathcal{H}_1$ is k_1-uniform, it is

$$\|\mathcal{H}_\tau\| = (2 + k_1)\|\mathcal{H}_2\|$$

and if $\mathcal{H}_2$ is k_2-uniform, one has

$$\|\mathcal{H}_\tau\| = (2m + \|\mathcal{H}_1\|)k_2$$

If both are uniform it follows

$$\|\mathcal{H}_\tau\| = (2 + k_1)mk_2$$

Reversely, assume the τ-independence of $\beta(\mathcal{H}_\tau)$, but suppose that none of $\mathcal{H}_1, \mathcal{H}_2$ is uniform. So, w.l.o.g. there are $b_j \in B_1$, $\hat{b}_j \in B_2$, $j \in [2]$, s.t. $|b_1| < |b_2|$ and $|\hat{b}_1| < |\hat{b}_2|$. Let $\tau_1, \tau_2 \in S_m$ be defined s.t. $\tau_1 = \tau_2$ restricted to $[m] \setminus \{1, 2\}$, and $\tau_1(j) = j$, $j \in [2]$, $\tau_2 = (1, 2)$, i.e., as the transposition. Hence

$$\begin{aligned} \beta(\mathcal{H}_{\tau_1}) - \beta(\mathcal{H}_{\tau_2}) &= \sum_{j \in [m]} |b_j| \left(|\hat{b}_{\tau_1(j)}| - |\hat{b}_{\tau_2(j)}|\right) \\ &= |b_1| \left(|\hat{b}_1| - |\hat{b}_2|\right) + |b_2| \left(|\hat{b}_2| - |\hat{b}_1|\right) \end{aligned}$$

So

$$\beta(\mathcal{H}_{\tau_1}) - \beta(\mathcal{H}_{\tau_2}) = (|b_2| - |b_1|) \left(|\hat{b}_2| - |\hat{b}_1|\right) > 0$$

yielding a contradiction. □

Corollary 1: For $m \in \mathbb{N}$, $\mathcal{H}_i$, $i \in [2]$, $\mathcal{H}_\tau$, τ as above

$$\omega(\mathcal{H}_\tau) \quad \textit{and} \quad \rho(\mathcal{H}_\tau) + \delta(\mathcal{H}_\tau)$$

are independent of τ iff one of $\mathcal{H}_1, \mathcal{H}_2$ is uniform.

PROOF. Since, for all $\mathcal{H} \in \mathfrak{H}$,

$$\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})} = 1 + \rho(\mathcal{H}) + \delta(\mathcal{H})$$

the assertion directly is implied by Prop. 4. □

Observe that (2), (4) of Thm. 1 are independent of $\tau \in S_m$. However, relying on (3) one obtains:

Proposition 5: For fixed integer m, and vertex-disjoint $\mathcal{H}_i \in \mathfrak{H}_0(V_i, m)$, $i \in [2]$, in general it depends on $\tau \in S_m$, whether $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2$ is diagonal or not.

PROOF. For $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, $m = 4$, let $\tau_1 = \mathrm{id}$, $\tau_2 = (1,2)(3,4) \in S_4$, and

$$\begin{aligned} B_1 &:= \{b_1 = x_1, b_2 = x_2, b_3 = x_3x_4, b_4 = x_5x_6\} \\ B_2 &:= \{\hat{b}_1 = y_1y_2, \hat{b}_2 = y_1y_3, \hat{b}_3 = y_4, \hat{b}_4 = y_5\} \end{aligned}$$

Then with $\mathcal{H}_\tau := \mathcal{H}_1 \otimes_\tau \mathcal{H}_2$ it is

$$\begin{aligned} B(\mathcal{H}_{\mathrm{id}}) = \{ & x_1y_1, x_1y_2, y_1y_2, x_2y_1, x_2y_3, y_1y_3, \\ & x_3x_4y_4, y_4, x_5x_6y_5, y_5\} \end{aligned}$$

implying $\mathcal{H}_{\mathrm{id}} \in \mathfrak{H}_{\mathrm{diag}}$, because it contains the y_1-connection of two $\mathrm{FPP}(1)$ according to La. 4 (i) in [16]. Whereas

$$\begin{aligned} B(\mathcal{H}_{\tau_2}) = \{ & x_1y_4, y_4, x_2y_5, y_5, x_3x_4y_1, x_3x_4y_2, \\ & y_1y_2, x_5x_6y_1, x_5x_6y_3, y_1y_3\} \end{aligned}$$

yields $\mathcal{H}_{\tau_2} \in \mathfrak{H}_0$ due to Thm. 1 (3). □

Before treating the case, that one of $\mathcal{H}_1, \mathcal{H}_2$ is diagonal, we state a useful fact:

Proposition 6: Let $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, s.t. $V_2 \subsetneq V_1$, $B_1 \cap B_2 = \emptyset$, $\mathcal{H}_1 \in \mathfrak{H}_0$ and $\mathcal{H}_2 \in \mathfrak{H}_{\mathrm{diag}}$. Let $\alpha_i := \alpha(\mathcal{H}_i)$, $i \in [2]$, $\alpha \in \{\beta, \omega, \rho, \delta\}$.

(1) *One has*

$$\begin{aligned} \omega(\mathcal{H}_1 \cup \mathcal{H}_2) &= \omega_1\omega_2 2^{|V_2|} \\ \delta(\mathcal{H}_1 \cup \mathcal{H}_2) &= \omega_1\delta_2 2^{|V_2|} + \sum_{p \in [\rho_2+1]} \varphi_p \end{aligned}$$

where $\varphi_p \in [\omega_1 2^{|V_2|}]_0$ denotes the number of orbits of $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \cup \mathcal{H}_2)$, whose representing transversals have restrictions to B_2, all lying in the pth satisfiable G_{V_2}-orbit of $\mathcal{F}(\mathcal{H}_2)$. Further it is

$$\sum_{p \in [\rho_2+1]} \varphi_p = (1+\rho_2)\omega_1 2^{|V_2|} - (1 + \rho(\mathcal{H}_1 \cup \mathcal{H}_2))$$

(2) *There exists $p \in [\rho_2 + 1]$ with $\varphi_p > 0$, if there is $\mathcal{H}_2' \subsetneq \mathcal{H}_2$ with $\mathcal{H}_2' \in \mathfrak{H}_0$ s.t. $\mathcal{H}_1 \cup \mathcal{H}_2' \in \mathfrak{H}_{\mathrm{diag}}$. In particular, one has $\varphi_p > 0$, for all $p \in [\rho_2+1]$, if there is $\mathcal{H}_2' \subsetneq \mathcal{H}_2$ with $\beta(\mathcal{H}_2') = 0$ s.t. $\mathcal{H}_1 \cup \mathcal{H}_2' \in \mathfrak{H}_{\mathrm{diag}}$.*

(3) *If $\varphi_p =: \varphi \in \mathbb{N}_0$, for all $p \in [\rho_2 + 1]$, then*

$$\varphi = \omega_1 2^{|V_2|} - \lambda$$

where

$$\lambda := (1 + \rho(\mathcal{H}_1 \cup \mathcal{H}_2))/(1 + \rho_2) \in \mathbb{N}$$

with $\lambda \le \omega_1 2^{|V_2|}$, and $\lambda = \omega_1 2^{|V_2|}$ iff $\varphi = 0$.

(4) *If the incidence graph of $\mathcal{H}_1$ admits a matching M covering the B_1-component, i.e. $M(B_1) = B_1$, s.t. $V_2 \cap M(V_1) = \emptyset$, then*

$$\delta(\mathcal{H}_1 \cup \mathcal{H}_2) = \omega_1\delta_2 2^{|V_2|}$$

PROOF. Since by assumption $B_1 \cap B_2 = \emptyset$, $V_1 \cup V_2 = V_1$, it is

$$\beta(\mathcal{H}_1 \cup \mathcal{H}_2) = \sum_{b \in B_1 \cup B_2} |b| - |V_1| = \beta_1 + \beta_2 + |V_2|$$

so the assertion for $\omega(\mathcal{H}_1 \cup \mathcal{H}_2)$ is true.

Every diagonal orbit over $\mathcal{H}_2$ has $2^{|V_2|}$ members, each of which yields a distinct diagonal orbit of $\mathcal{F}(\mathcal{H}_1 \cup \mathcal{H}_2)$ when enlarged by a transversal over $\mathcal{H}_1$ representing a G_{V_1}-orbit. Thus one obtains the contribution $\omega_1\delta_2 2^{|V_2|}$ to $\delta(\mathcal{H}_1 \cup \mathcal{H}_2)$, and the assertion for $\delta(\mathcal{H}_1 \cup \mathcal{H}_2)$ directly follows from the definition of φ_p, $p \in [\rho_2+1]$. The last assertion in (1) directly is implied by using twice the general formula $\omega - (1+\rho) = \delta$ valid for every BHG.

Regarding (2), assume that there is $\mathcal{H}_2' = (V_2', B_2') \subsetneq \mathcal{H}_2$ with $\mathcal{H}_2' \in \mathfrak{H}_0$ s.t. $\mathcal{H}_1 \cup \mathcal{H}_2' \in \mathfrak{H}_{\mathrm{diag}}$. For fixed $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \cup \mathcal{H}_2')$, let F_2' be its satisfiable restriction to B_2', so it has a model, say w_2' over V_2'. Set $\mathcal{H}_2'' = (V_2'', B_2'') := \mathcal{H}_2 \setminus \mathcal{H}_2'$. Then F_2' can be extended over B_2'' to a satisfiable transversal F_2 over $\mathcal{H}_2$ by letting each literal over $x \in V_2'' \setminus V_2'$ be pure in F_2, and each literal over $x \in V_2'' \cap V_2'$ be defined according to w_2'. Thus there is $p \in [\rho_2 + 1]$ with $\varphi_p > 0$.

Next, $\beta(\mathcal{H}_2') = 0$ implies $\mathcal{H}_2' \in \mathfrak{H}_0$ and $\mathcal{F}(\mathcal{H}_2') = \mathcal{F}_{\mathrm{comp}}(\mathcal{H}_2')$. Hence, there is only one orbit of transversals over $\mathcal{H}_2'$. Choose a representative F_2' of this orbit with the property that $F_1 \cup F_2' \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \cup \mathcal{H}_2')$, for an appropriate $F_1 \in \mathcal{F}(\mathcal{H}_1)$. Clearly, any satisfiable extension F_2 of F_2' over B_2'' yields $F_1 \cup F_2 \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \cup \mathcal{H}_2)$. Thus, $\varphi_p > 0$, for all $p \in [\rho_2 + 1]$, so (2) is verified.

If $\varphi_p =: \varphi \in \mathbb{N}_0$, for all $p \in [\rho_2 + 1]$, it is

$$\sum_{p \in [\rho_2+1]} \varphi_p = (\rho_2 + 1)\varphi = (1 + \rho_2)\left(\omega_1 2^{|V_2|} - \lambda\right)$$

using (1), implying

$$\lambda = (1 + \rho(\mathcal{H}_1 \cup \mathcal{H}_2))/(1 + \rho_2) \in \mathbb{N}$$

Since $\sum_{p \in [\rho_2+1]} \varphi_p \ge 0$, one obtains $\lambda \le \omega_1 2^{|V_2|}$, and $\lambda = \omega_1 2^{|V_2|}$ iff $\varphi = 0$, so (3).

Concerning (4), observe that by assumption every transversal over $\mathcal{H}_1$ can be satisfied independently by the variables in $M(V_1)$. So one has $\varphi_p = 0$, for all $p \in [\rho_2+1]$, because every transversal in $\mathcal{F}(\mathcal{H}_1 \cup \mathcal{H}_2)$ having a satisfiable restriction to B_2 then is satisfiable. □

Remark 2: 1. There might be cases where

$$\varphi_p = \varphi \in [\omega_1 2^{|V_2|}]_0$$

is a constant, for all $p \in [\rho_2 + 1]$, confer e.g. the proof of Prop. 7 below. But in general that is untrue: Let $\mathcal{H}_2 = (V_2, B_2)$ with

$$B_2 = \{xy_1, xy_2, y_1y_2, xy_3, xy_4, y_3y_4\}$$

which is minimal diagonal due to La. 4 (i) in [16]. Let $\mathcal{H}_1 = (V_1, B_1)$ with

$$B_1 = \{xy_2y_4z, y_1y_3\}$$

which is trivial and $\omega_1 = 1$. Let $F_1 := B_1 \in \mathcal{F}_{\mathrm{comp}}$ be the unique positive monotone representing transversal. Any $F_2 \in \mathcal{F}(\mathcal{H}_2)$, in which x appears as a pure literal, is satisfiable because setting x accordingly remains two clauses which can be solved always. Moreover, extending F_2 by F_1, yields a satisfiable transversal over $\mathcal{H}_1 \cup \mathcal{H}_2$, because e.g. the variables y_1, y_2, y_4, z can be assigned independently to solve the remaining clauses over $y_1y_3, y_1y_2, y_3y_4, xy_2y_4z$, respectively. Let the orbit represented by F_2 have the number $p \in [\rho_2 + 1]$. As all its members have x as a pure literal, one obtains $\varphi_p = 0$. Let

$$F_2' = \{x\bar{y}_1, \bar{x}y_2, \bar{y}_2\bar{y}_1, x\bar{y}_3, \bar{x}y_4, \bar{y}_3\bar{y}_4\}$$

which is satisfied via $x = y_1 = y_3 = 0$, where y_1, y_3 have unique satisfying assignments in every model. Hence $F_1 \cup$

$F_2' \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \cup \mathcal{H}_2)$. Further, x admits no pure literal in F_2', so its orbit of number $p' \neq p$ yields $\varphi_{p'} > 0$.

2. Note that the converse of statement (4) in general is false: Simply, choose any $\mathcal{H}_1 \in \mathfrak{H}_0$ that does not meet the matching condition, but whose retraction $\mathcal{H}_1[V_1 \setminus V_2] \in \mathfrak{H}_0$, so $\varphi_p = 0$, for all $p \in [\rho_2+1]$. To present a concrete example, first let $\mathcal{H}_2 \in \mathcal{H}_{\mathrm{diag}}$ be the complete BHG with vertex set $\{x, y\}$. Next, refering to La. 4 (ii) in [16] and its proof, it is $\mathcal{H} := (V, B) \in \mathfrak{H}_{\mathrm{mdiag}}$ where

$$B := \{uv, up, vp, vq, pq, uq\}$$

Thus $\mathcal{H}' := \mathcal{H} \setminus \{vq\} \in \mathfrak{H}_0$. Let $\mathcal{H}_1$ be obtained from $\mathcal{H}'$ by enlarging, say $\{uv\}$ about x and $\{pq\}$ about y ensuring $V_2 \subsetneq V_1$ and $\mathcal{H}' = \mathcal{H}_1[V_1 \setminus V_2] \in \mathfrak{H}_0$. There obviously is a matching of the incidence graph of $\mathcal{H}_1$ using all variables covering its B_1-component and establishing $\mathcal{H}_1 \in \mathcal{H}_0$. But there is no such matching relying on members of $V_1 \setminus V_2$, only, as required in Prop. 6 (4).

Returning to the τ-dependence of δ one has:

Proposition 7: For $m \in \mathbb{N}$, let $\mathcal{H}_i \in \mathfrak{H}(V_i, m)$ be non-uniform, $i \in [2]$, where $V_1 \cap V_2 = \emptyset$. If one of $\mathcal{H}_1, \mathcal{H}_2$, is diagonal, the values of $\delta(\mathcal{H}_\tau)$, $\rho(\mathcal{H}_\tau) \in \mathbb{N}$ can depend on $\tau \in S_m$, where $\mathcal{H}_\tau := \mathcal{H}_1 \otimes_\tau \mathcal{H}_2$.

PROOF. It suffices to verify the assertion for δ. So let $m = 3$, $\mathcal{H}_i =: (V_i, B_i)$, $i \in [2]$, with

$$\begin{aligned} B_1 &:= \{u, w, uv\} =: \{b_1, b_2, b_3\} \\ B_2 &:= \{x, y, xy\} =: \{\hat{b}_1, \hat{b}_2, \hat{b}_3\} \end{aligned}$$

and $V := V_1 \cup V_2$. Hence $\mathcal{H}_2 = \mathcal{K}_2 \in \mathfrak{H}_{\mathrm{simp}}$ is diagonal. Defining $\sigma := (1,3)$, $\tau := (1) \in S_3$ one obtains

$$\mathcal{H}_\psi = \mathcal{H}_0 \cup \mathcal{H}_2 \cup \{b_\psi\}, \quad \psi \in \{\sigma, \tau\}$$

where

$$B(\mathcal{H}_0) := \{ux, wy, uvx\}, b_\sigma := \{uy\}, b_\tau := \{uvy\}$$

Thus

$$\omega(\mathcal{H}_0 \cup \{b_\sigma\}) = 2^4, \quad \omega(\mathcal{H}_0 \cup \{b_\tau\}) = 2^5$$

The incidence graph of $\mathcal{H}_0 \cup \{b_\psi\}$ admits a matching M_ψ, which contains the vertices $\{u, w, v, y\}$ covering all edges of $\mathcal{H}_0 \cup \{b_\psi\}$, which therefore is a non-diagonal BHG that has no hyperedge in common with B_2, $\psi \in \{\sigma, \tau\}$. Clearly $\delta(\mathcal{H}_2) = 1$, $\rho(\mathcal{H}_2) = 3$, and $V_2 \subsetneq V(\mathcal{H}_0)$. Thus using Prop. 6 (1), it follows

$$\begin{aligned} \delta(\mathcal{H}_\psi) &= \delta(\mathcal{H}_0 \cup \{b_\psi\} \cup \mathcal{H}_2) \\ &= \omega(\mathcal{H}_0 \cup \{b_\psi\})2^2 + \sum_{p \in [3]} \varphi_p(\psi) \end{aligned}$$

for $\psi \in \{\sigma, \tau\}$.

Let $F_0 := \{ux, wy\}$. First regarding $\psi = \sigma$, the orbits of $\mathcal{F}(\mathcal{H}_0 \cup \{b_\sigma\})$ may be represented by the transversals as follows:

$$\begin{aligned} F_{1,l} &= F_0 \cup \{uvx\} \cup \{c_{\sigma,l}\} \\ F_{2,l} &= F_0 \cup \{\bar{u}vx\} \cup \{c_{\sigma,l}\} \\ F_{3,l} &= F_0 \cup \{uv\bar{x}\} \cup \{c_{\sigma,l}\} \\ F_{4,l} &= F_0 \cup \{\bar{u}v\bar{x}\} \cup \{c_{\sigma,l}\} \end{aligned}$$

for all $l \in [4]$, with

$$c_{\sigma,l} \in W_{b_\sigma} = \{uy, \bar{u}y, u\bar{y}, \bar{u}\bar{y}\}$$

where W_{b_σ} is assumed to have that fixed ordering. Evidently $\{wy\}$, as well as the clauses over $\{uvx\}$, can be solved independently via $w = v = 1$. If $x = 1$ is possible, then all clauses of $F_{j,l}$ are solved, $j, l \in [4]$. So, only the case that x is forced to 0 meaning $u = 1$, leads to an unsatisfiable transversal iff $c_{\sigma,l} \in F_{j,l}$ contains $\bar{u}$, which is the case for $l = 2, 4$, and simultaneously its literal over y is forced to 0. This situation can occur only for the partial $\mathcal{H}_2$-patterns $\bar{x}, y$, as well as $\bar{x}, \bar{y}$. Each of these patterns occurs exactly once in all three satisfiable $\mathcal{H}_2$-orbits. It is not hard to verify, that $F_{j,l}$, for all $j \in [4]$ and $l \in \{2, 4\}$, becomes unsatisfiable by exactly one of these patterns, thus $\varphi_p(\sigma) = 8$, for every $p \in [3]$. Therefore

$$\delta(\mathcal{H}_\sigma) = 2^4 2^2 + 3 \cdot 8 = 88$$

The representatives of the orbits of $\mathcal{F}(\mathcal{H}_0 \cup \{b_\tau\})$ are obtained from those $F_{j,l}$, $j, l \in [4]$, as above, by doubling each, providing two instances. Then, enlarging the clause $c_{\sigma,l}$ in its first instance about v yielding $F^v_{j,l}$ containing the modified $c^v_{\tau,l}$, and in its second instance about $\bar{v}$ yielding $F^{\bar{v}}_{j,l}$ containing $c^{\bar{v}}_{\tau,l}$, for all $j, l \in [4]$. Observe that the transversals $F^v_{j,l}$, $j, l \in [4]$, are satisfiable independently of V_2, because $\{ux\}$, $\{wy\}$ and all the clauses over $\{uvx\}$, as well as $c^v_{\tau,l}$, are solved via $u = w = v = 1$. Any $F^{\bar{v}}_{j,l}$ can become unsatisfiable only, if x is forced to 0 meaning $u = 1$, which is critical only for $j = 2, 4$ forcing $v = 1$. Hence $c^{\bar{v}}_{\tau,l}$, for $l = 2, 4$, is unsatisfiable iff y is forced to $0, 1$, respectively. That is possible only for the partial $\mathcal{H}_2$-patterns $\bar{x}, \bar{y}$, for $l = 2$, respectively $\bar{x}, y$, for $l = 4$. So, $\varphi_p(\tau) = 4$, for every $p \in [3]$, yielding

$$\delta(\mathcal{H}_\tau) = 2^5 2^2 + 3 \cdot 4 = 140 > \delta(\mathcal{H}_\sigma) \qquad \square$$

VI. A Commutative Extension of the Joining Map

For every variable-disjoint members $C_i \in \mathrm{CNF}(V_i, m)$, $i \in [2]$, each regarded as fixed labeled over $[m]$, we set $C_1 \boxtimes C_2 := (C_1 \otimes_{\mathrm{id}} C_2) \cup (C_2 \otimes_{\mathrm{id}} C_1)$, for id $\in S_m$, called the *symmetric join* of C_1, C_2. This yields a commutative operation, for $C_1 = \{c_1, \ldots, c_m\}$, $C_2 = \{\hat{c}_1, \ldots, \hat{c}_m\}$, because by definition,

$$C_1 \boxtimes C_2 = \bigcup_{j \in [m]} c_j \otimes \hat{c}_j \cup \bigcup_{j \in [m]} \hat{c}_j \otimes c_j = C_2 \boxtimes C_1$$

Evidently, $C_1 \boxtimes C_2 = \bigcup_{j \in [m]} c_j \boxtimes \hat{c}_j$ with $c_j \boxtimes \hat{c}_j := (c_j \otimes \hat{c}_j) \cup (\hat{c}_j \otimes c_j)$. The symmetric join is independent of the labeling in the sense that given $\sigma \in S_m$ and $\{c_{\sigma(i)} : i \in [m]\}$, $\{\hat{c}_{\sigma(i)} : i \in [m]\}$, reordered C_1^σ, C_2^σ are obtained with $C_1^\sigma \boxtimes C_2^\sigma = C_1 \boxtimes C_2$. In general, that is false if C_1^π, C_2^σ are joined symmetrically, for distinct $\pi, \sigma \in S_m$. Analogously, for vertex-disjoint $\mathcal{H}_i \in \mathfrak{H}(V_i, m)$, $i \in [2]$, with fixed labeled edge sets, we set $\mathcal{H}_1 \boxtimes \mathcal{H}_2 := (V_1 \cup V_2, B(B(\mathcal{H}_1) \boxtimes B(\mathcal{H}_2)))$.

More generally, one defines the following mapping:

Definition 2: On basis of $\mathrm{CNF}(V_1, m) \uplus \mathrm{CNF}(V_2, m) := [\mathrm{CNF}(V_1, m) \times \mathrm{CNF}(V_2, m)] \cup [\mathrm{CNF}(V_2, m) \times \mathrm{CNF}(V_1, m)]$ *with $V_1 \cap V_2 = \emptyset$ set*

$$\boxtimes_m : [\mathrm{CNF}(V_1, m) \uplus \mathrm{CNF}(V_2, m)] \times S_m \to \mathrm{CNF}$$

Here among $(C, \hat{C}) \in \mathrm{CNF}(V_1, m) \uplus \mathrm{CNF}(V_2, m)$ exactly one, say C, is a member of $\mathrm{CNF}(V_1, m)$. Then with $C = \{c_j : j \in [m]\}$, $\hat{C} = \{\hat{c} : j \in [m]\} \in \mathrm{CNF}(V_2, m)$ it is

$$\boxtimes_m(C, \hat{C}, \sigma) := \bigotimes_m^{V_1, V_2}(C, \hat{C}, \sigma) \cup \bigotimes_m^{V_2, V_1}(\hat{C}, C, \sigma^{-1})$$

$$= \bigcup_{j\in[m]} c_j \otimes \hat{c}_{\sigma(j)} \cup \bigcup_{j\in[m]} \hat{c}_j \otimes c_{\sigma^{-1}(j)} =: C \boxtimes_\sigma \hat{C}$$

which is called the symmetric (σ-)join *of* $C,\hat{C}$, *for* $\sigma \in S_m$. *We set* $\boxtimes_{\mathrm{id}} =: \boxtimes$.

The symmetric σ-join obviously is commutative in C, $\hat{C}$ independent of σ, but it never yields an associative operation. An analogous mapping is defined for BHGs as follows:

Definition 3: For $\mathfrak{H}(V_1, m) \uplus \mathfrak{H}(V_2, m) := [\mathfrak{H}(V_1, m) \times \mathfrak{H}(V_2, m)] \cup [\mathfrak{H}(V_2, m) \times \mathfrak{H}(V_1, m)]$, *with* $V_1 \cap V_2 = \emptyset$, *set*

$$\boxtimes_m^{\mathfrak{H}} : [\mathfrak{H}(V_1, m) \uplus \mathfrak{H}(V_2, m)] \times S_m \to \mathfrak{H}$$

where, for $(\mathcal{H}_1, \mathcal{H}_2) \in \mathfrak{H}(V_1, m) \uplus \mathfrak{H}(V_2, m)$ *with* $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, $\tau \in S_m$, *their* symmetric τ-join *is*

$$\boxtimes_m^{\mathfrak{H}}(\mathcal{H}_1, \mathcal{H}_2, \tau) := \left(V_1 \cup V_2, B\left(\boxtimes_m(B_1, B_2, \tau)\right)\right)$$

which is abbreviated as

$$\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2 = (\mathcal{H}_1 \otimes_\tau \mathcal{H}_2) \cup (\mathcal{H}_2 \otimes_{\tau^{-1}} \mathcal{H}_1)$$

For simplifying the notation, set $\mathrm{CNF}(m) := \{C \in \mathrm{CNF} : |C| = m\}$, and $\mathfrak{H}(m) := \{\mathcal{H} \in \mathfrak{H} : |B(\mathcal{H})| = m\}$.

Theorem 3: Let $m \in \mathbb{N}$, $C = \{c_j : j \in [m]\}, D = \{d_j : j \in [m]\} \in \mathrm{CNF}(m)$ *be fixed labeled over* $[m]$ *with* $V(C) \cap V(D) = \emptyset$, $\sigma \in S_m$, *then the following assertions are true for* $J := C \boxtimes_\sigma D$:

(1) *Let* m_1 *be the number of indices* $j \in [m]$, *s.t.* $|c_j| = |d_j| = 1$, *then*

$$\begin{aligned} |J| &= \|C\| + \|D\| - m_1 + 2m \\ \|J\|/2 &= \|C\| + \|D\| - m_1 + \sum_{j\in[m]} |c_j| \cdot |d_j| \end{aligned}$$

(2) $J \in \mathrm{SAT}$ *iff* $\mathcal{M}(C \cup C^\gamma) \neq \emptyset \neq \mathcal{M}(D \cup D^\gamma)$.

(3) $J \in \mathcal{I}$ *iff* $m = 1$ *and* C, D *contain unit clauses only.*

(4) J *is a transversal iff* C, D *are transversals.*

(5) J *is linear iff* C, D *consist of non-complementary unit clauses only.*

(6) J *never is exact linear.*

(7) $\mathcal{H}(J)$ *never is uniform.*

(8) *Let* C, D *be* k_C-, k_D*-uniform transversals, respectively. Then* $\mathcal{H}(J)$ *is* r*-regular iff* $(2+k_C)|r$, $(2+k_D)|r$, $\mathcal{H}(C)$, $\mathcal{H}(D)$ *are regular, and*

$$r = (2 + k_C)\deg(\mathcal{H}(D)) = (2 + k_D)\deg(\mathcal{H}(C))$$

(9) *Assume there is* $m' \in \mathbb{N}$, $\tau \in S_{m'}$ *s.t.* $\mathcal{H}(C), \mathcal{H}(D) \in \mathfrak{H}(m')$, *and that* $C \otimes_\sigma D$ *is fibre-respecting w.r.t.* τ, $D \otimes_{\sigma^{-1}} C$ *is fibre-respecting w.r.t.* τ^{-1}. *Then* $\mathcal{H}(J) = \mathcal{H}(C) \boxtimes_\tau \mathcal{H}(D)$. *That is valid in particular if* C, D *are transversals equally labeled as their BHGs, and* $\sigma = \tau$.

PROOF. By the definition of the symmetric σ-join

$$Q := (C \otimes_\sigma D) \cap (D \otimes_{\sigma^{-1}} C)$$

consists of exactly those $t \in J$, for which there is $j \in [m]$, $c_j \in C$, $d_{\sigma(j)} \in D$, s.t.

$$t \in \left(c_j \otimes d_{\sigma(j)}\right) \cap \left(d_{\sigma(j)} \otimes c_j\right)$$

So, there must be $l^c \in c_j$, $l^d \in d_{\sigma(j)}$ with

$$t = c_j \cup \{l^d\} = d_{\sigma(j)} \cup \{l^c\}$$

As $V(c_j) \cap V(d_{\sigma(j)}) = \emptyset$, $t \in Q$ is equivalent with

$$c_j = \{l^c\}, \quad d_{\sigma(j)} = \{l^d\}, \quad t = \{l^c, l^d\}$$

Therefore it is $|Q| = m_1$ and $\|Q\| = 2m_1$ implying (1) on basis of Prop. 13 (2) in [16].

Let $J \in \mathrm{SAT}$, and suppose that $\mathcal{M}(C) \cap \mathcal{M}(C^\gamma) = \emptyset$. If $w \in \mathcal{M}(J)$, then w satisfies $C^\gamma \subseteq J$. By assumption $w \notin \mathcal{M}(C)$, so there is $j \in [m]$ s.t. $w(V(c_j)) = c_j^\gamma$. Thus

$$c_j \otimes d_{\sigma(j)} = \left\{c_j \cup \{l\} : l \in d_{\sigma(j)}\right\} \cup \left\{d_{\sigma(j)}^\gamma\right\} \subseteq J$$

remains unsatisfied providing a contradiction.

Conversely, let

$$w_C \in \mathcal{M}(C) \cap \mathcal{M}(C^\gamma), \quad w_D \in \mathcal{M}(D) \cap \mathcal{M}(D^\gamma)$$

then it is claimed that $w := w_C \cup w_D \in \mathcal{M}(J)$: Evidently, w then satisfies $C^\gamma \cup D^\gamma \subseteq J$. Each remaining $t \in J$ either contains a clause of C or a clause of D, thus is satisfied by w, and (2) is proven.

Regarding the sufficiency of (3), for variables x, y, let $C = \{l(x)\}, D = \{l(y)\}$, then

$$J = \left\{l(x)l(y), \bar{l}(x), \bar{l}(y)\right\} \in \mathcal{I}$$

For the necessity, we claim that if one of $C \cup C^\gamma, D \cup D^\gamma$, say $C \cup C^\gamma$, is unsatisfiable then

$$J' := (C \otimes_\sigma D) \cup C^\gamma \in \mathrm{UNSAT}$$

From this claim (3) follows: If $J' \subsetneq J$ as a proper subformula one has $J \notin \mathcal{I}$. So, $J \in \mathcal{I}$ implies $J = J'$. That is equivalent with $(D \otimes_{\sigma^{-1}} C) \setminus C^\gamma \subseteq J'$, which is equivalent with

$$Q = (D \otimes_{\sigma^{-1}} C) \setminus C^\gamma = \left\{l_j^c l_j^d : j \in [m]\right\}$$

refering to the proof of (1). Hence, each clause in C, D has to be unit, so $Q = (C \otimes_\sigma D) \setminus D^\gamma$. Since $J \in \mathcal{I}$ is connected, one concludes $m = 1$.

It remains to verify the claim: W.l.o.g. let $C \cup C^\gamma \in \mathrm{UNSAT}$, then either $C \in \mathrm{UNSAT}$ meaning $C^\gamma \in \mathrm{UNSAT}$, so $J' \in \mathrm{UNSAT}$. Or $C, C^\gamma \in \mathrm{SAT}$, then supposing $J' \in \mathrm{SAT}$, every $w \in \mathcal{M}(J')$ satisfies C^γ. Since $C \cup C^\gamma \in \mathrm{UNSAT}$, there must be a clause $c_j \in C$ with $w(V(c_j)) = c_j^\gamma$. Hence, all clauses in

$$(c_j \otimes d_{\sigma(j)}) \setminus \left\{d_{\sigma(j)}^\gamma\right\} \subseteq J'$$

are forced to be satisfied via $w(V(d_{\sigma(j)})) = d_{\sigma(j)}$, thus $d_{\sigma(j)}^\gamma \in J'$ remains unsatisfied providing a contradiction. So, the claim is true yielding (3).

The necessity assertion in (4) is true by Prop. 13 (5) in [16]. Reversely, let C, D be transversals, then by the same result $C \otimes_\sigma D$ and $D \otimes_{\sigma^{-1}} C$ are transversals. Suppose there is $b \in B(J)$ with $t, t' \in J_b$ meaning

$$b \in B(C \otimes_\sigma D) \cap B(D \otimes_{\sigma^{-1}} C)$$

Thus, there are $x \in V(C), y \in V(D)$ with

$$b \in (\{x\} \otimes \{y\}) \cap (\{y\} \otimes \{x\}) = \{xy\}$$

By the definition of the symmetric join, one obtains $t = t'$, as shown in the proof of (1), so J is a transversal.

If J is linear and one of C, D contains a clause with 2 variables, say x, y, then by definition J contains 2 clauses in which x, y appear, contradicting its linearity. As $C^\gamma, D^\gamma \subset$

J, they have to be linear, so do not contain complementary unit clauses. The reverse direction is obvious, providing (5).

Since $m \geq 1$, and $C^\gamma, D^\gamma \subseteq J$ which are variable-disjoint, (6) is true. From Prop. 13 (8) in [16], one concludes that the k-uniformity of $\mathcal{H}(J)$ means, that $\mathcal{H}(C)$ simultaneously has to be $(k-1)$-, and $(k+1)$-uniform, yielding (7).

Addressing (8), with C, D also J is a transversal using (4). On basis of the uniformity of C, D, the regularity of J is equivalent with $|J(x)| = (2+k_D)|C(x)| = r$, for all $x \in V(C)$, and $|J(x)| = (2+k_C)|D(x)| = r$, for all $x \in V(D)$. Thus

$$\begin{aligned} |C(x)| &= \deg(\mathcal{H}(C)) = r/(2+k_D) \\ |D(x)| &= \deg(\mathcal{H}(D)) = r/(2+k_C) \end{aligned}$$

providing the equivalence with

$$r = (2+k_C)\deg(\mathcal{H}(D)) = (2+k_D)\deg(\mathcal{H}(C))$$

To verify (9), one derives

$$\begin{aligned} \mathcal{H}(J) &= \mathcal{H}((C \otimes_\sigma D) \cup (D \otimes_{\sigma^{-1}} C)) \\ &= \mathcal{H}(C \otimes_\sigma D) \cup \mathcal{H}(D \otimes_{\sigma^{-1}} C) \\ &= (\mathcal{H}(C) \otimes_\tau \mathcal{H}(D)) \cup (\mathcal{H}(D) \otimes_{\tau^{-1}} \mathcal{H}(C)) \\ &= \mathcal{H}(C) \boxtimes_\tau \mathcal{H}(D) \end{aligned}$$

where Prop. 3 has been used. □

Remark 3: 1. Observe that assertion (2) in Thm. 3 is equivalent with $J \in$ SAT iff $C, D \in$ NAESAT.

2. For $C \in \mathcal{S} \cap \mathcal{I}$, which in particular is true in case of a fibre-formula $C = W_{V(C)}$, one has $C = C^\gamma$, so the first condition of (2) is invalid. This situation can occur for a non-fibre formula also. Consider e.g.

$$C = \{xy, yz, zx, \bar{x}\bar{y}, \bar{y}\bar{z}, \bar{z}\bar{x}\} \in \mathcal{S}$$

Here $\mathcal{H}(C)$ is 2-uniform and a triangle, hence there is no proper 2-coloring of its vertex set, so $C \in$ UNSAT; its membership to $\mathcal{I}$ can be verified easily.

3. The assumption in assertion (9) concretely means that there are labelings

$$\mathcal{H}(C) = \{b_j : j \in [m']\}, \quad \mathcal{H}(D) = \{\hat{b}_j : j \in [m']\}$$

s.t. $|C_{b_j}| = |D_{\hat{b}_{\tau(j)}}|$, and that the clauses in $C_{b_j}, D_{\hat{b}_{\tau(j)}}$ are labeled consecutively, for all $j \in [m']$, respectively.

Concerning the second paragraph of Rem. 3, one has:

Proposition 8: Let $\mathcal{H}_1, \mathcal{H}_2$ be vertex-disjoint, of equal size m, then the following assertions are true, for every $\tau \in S_m$:

1. *$\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2 \in$ BIC iff $\mathcal{H}_1, \mathcal{H}_2 \in$ BIC.*
2. *$\mathcal{H}_1 \otimes_\tau \mathcal{H}_2 \in$ BIC iff $\mathcal{H}_1, \mathcal{H}_2 \in$ BIC.*

PROOF. Let $\mathcal{H}_1, \mathcal{H}_2 \in$ BIC, then by the definition of the corresponding join-operations $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2, \mathcal{H}_1 \boxtimes \mathcal{H}_2 \in$ BIC, for every fixed $\tau \in S_m$. As $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}_1 \boxtimes \mathcal{H}_2$, the reverse of (1) is valid.

Finally, let $\mathcal{H}_\tau = (V, B) := \mathcal{H}_1 \otimes_\tau \mathcal{H}_2 \in$ BIC, for $\tau \in S_m$, then $\mathcal{H}_2 \in$ BIC as a sub-hypergraph. Suppose $\mathcal{H}_1 \notin$ BIC, then for any 2-coloring of V, there is $b \in B(\mathcal{H}_1)$ appearing monochromatic. Thus, $\mathcal{H}(b \otimes \tilde{\tau}(b)) \subseteq \mathcal{H}_\tau$ contains a monochromatic edge, namely $\tilde{\tau}(b)$, providing (2). □

Lemma 2: For $\mathcal{H} = (V, B) \in \mathfrak{H}$ one has:

(i) *An orbit $\mathcal{O} \in \mathcal{F}(\mathcal{H})/G_V$ is contained in NAESAT iff its representative lies in NAESAT.*

(ii) *$\mathcal{F}_{\mathrm{comp}}(\mathcal{H}) \subseteq$ NAESAT iff $\mathcal{H} \in$ BIC.*

(iii) *$\mathcal{F}_{\mathrm{comp}}(\mathcal{H}) \subseteq$ NAESAT if $\beta(\mathcal{H}) = 0$ and $\mathcal{H}$ is loopless, or $\mathcal{H}$ is 2-uniform and bipartite as a simple graph.*

(iv) *$\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \cap$ NAESAT $= \emptyset$.*

PROOF. Consider distinct $F, F' \in \mathcal{O}$ s.t. $F \in$ NAESAT with model w. By transitivity and commutativity, there is a unique $X \in G_V$ s.t. $F^X = F'$, so $F^{\gamma X} = F^{X\gamma} = F'^\gamma$. Thus, w^X is a model of $(F \cup F^\gamma)^X = F' \cup F'^\gamma$ implying $F' \in$ NAESAT, so $\mathcal{O} \subseteq$ NAESAT. The reverse is evident, hence (i) is true.

$\mathcal{H}$ is bicolorable iff $B \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H}) \cap$ NAESAT as a monotone formula, so (ii) is implied by (i).

In both alternatives of (iii) it is $\mathcal{H} \in$ BIC, evidently. So (iii) is implied by (ii). Assertion (iv) is obvious. □

Lemma 3: Let $\mathcal{H}_i \in \mathfrak{H}(m)$, $i \in [2]$, $\tau \in S_m$, be vertex-disjoint with fixed labeled edge sets over $[m]$. Then $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2 \in \mathfrak{H}_0$ implies that $\mathcal{H}_i$ is loopless, $i \in [2]$.

PROOF. Let $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, $B_1 = \{b_j : j \in [m]\}$, $B_2 = \{\hat{b}_j : j \in [m]\}$. Suppose there is $j \in [m]$ s.t. w.l.o.g.

$$b_j = \{x\}, \quad \hat{b}_{\tau(j)} = \{y_1, \ldots, y_r\}$$

for fixed $r \in \mathbb{N}$. Then obviously

$$b_j \boxtimes \hat{b}_{\tau(j)} = \{\bar{x}, xy_1, \ldots, xy_r\} \cup \{\bar{y}_1 \cdots \bar{y}_r, y_1 \cdots y_r x\}$$

is a diagonal transversal over $\mathcal{H}(j) := \mathcal{H}(b_j \boxtimes \hat{b}_{\tau(j)})$. Since $\mathcal{H}(j) \subseteq \mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$, the assertion is established. □

In the following, for simplifying the notation, the discussion first is focused on the special case $\tau = \mathrm{id}$. In particular, for the symmetric join of trivial BHGs, one has:

Theorem 4: Let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(m)$, $\beta(\mathcal{H}_i) = 0$, $i \in [2]$, be vertex-disjoint with fixed labeled $B_1 = \{b_j : j \in [m]\}$, $B_2 = \{\hat{b}_j : j \in [m]\}$. Further setting $\mathcal{H} := \mathcal{H}_1 \boxtimes \mathcal{H}_2$, and $\mathcal{H}(j) := \mathcal{H}(b_j \boxtimes \hat{b}_j)$, $j \in [m]$, the following is true:

1. *$\mathcal{H} \in \mathfrak{H}_0$ iff $\mathcal{H}_1, \mathcal{H}_2$ are loopless.*
2. *$\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ iff $m = 1$ and $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}^\ell$. In this case, $\mathcal{H}$ is isomorphic to $\mathcal{K}_2$.*
3. *Let $\mathcal{H} \in \mathfrak{H}_0$ then*
   (a) *$\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$ iff there is $j \in [m]$, s.t. $\mathcal{H}(j) \in \mathfrak{H}_{\mathrm{maxnd}}$.*
   (b) *$\mathcal{H}$ is dense maximal non-diagonal iff $\mathcal{H}(j)$ is dense maximal non-diagonal, for all $j \in [m]$.*

PROOF. The necessity in (1) is provided by La. 3. Reversely, as $\beta(\mathcal{H}_1) = \beta(\mathcal{H}_2) = 0$, $\mathcal{H}$ is the vertex-disjoint union of the $\mathcal{H}(j)$ over all $j \in [m]$. Thus $\mathcal{H} \in \mathfrak{H}_0$, iff $\mathcal{H}(j) \in \mathfrak{H}_0$, for all $j \in [m]$, which is claimed to be true if $|b_j| > 1$, $|\hat{b}_j| > 1$, for all $j \in [m]$. To that end, for fixed $j \in [m]$, set $b := b_j$, $\hat{b} := \hat{b}_j$, $\mathcal{H}' := \mathcal{H}(j)$ which decomposes into the edge-disjoint parts $\mathcal{H}'_I$, $\mathcal{H}'_{II}$. Here $\mathcal{H}'_I$ is obtained from $\mathcal{H}(b \otimes \hat{b})$ via substituting $\hat{b}$ by b; and $\mathcal{H}'_{II}$ arises from $\mathcal{H}(\hat{b} \otimes b)$ via substituting b by $\hat{b}$. So, for $F \in \mathcal{F}(\mathcal{H}')$, one has

$$F_I \in \mathcal{F}(\mathcal{H}'_I),\ F_{II} \in \mathcal{F}(\mathcal{H}'_{II}), \quad \text{s.t.} \quad F = F_I \cup F_{II}$$

as disjoint union. For verifying that $F \in$ SAT, we show that F_I, F_{II} can always be satisfied. Observing that $F_I[b] \subseteq W_b$ and $F_{II}[\hat{b}] \subseteq W_{\hat{b}}$, we distinguish three cases for these retractions.

Case (i): $F_I[b] \subsetneq W_b$, $F_{II}[\hat{b}] \subsetneq W_{\hat{b}}$, then both are satisfiable only relying on the variables in $b, \hat{b}$, respectively. So, $F_I(b) = F_I, F_{II}(\hat{b}) = F_{II} \in$ SAT implying $F \in$ SAT.

Case (ii): W.l.o.g.

$$F_I[b] = W_b, \quad F_{II}[\hat{b}] \subsetneq W_{\hat{b}}$$

Let $c' \in F_I \cap W_b$, which is the only clause therein. Given $c \in F_I[b]$, let m_c be the number of clauses in F_I containing c, and set

$$\tilde{m} := \min\{m_c : c \in F_I[b] \setminus \{c'\}\} \geq 1$$

thus

$$|\hat{b}| = |F_I \setminus \{c'\}| \geq (2^{|b|} - 1)\tilde{m} + m_{c'} - 1$$

Fix $\tilde{c} \in F_I[b]$ distinct to c' s.t. $m_{\tilde{c}} = \tilde{m}$. Due to La. 1 (v) in [16], as $F_I[b] \in \mathcal{I}$, there is an assignment $\tilde{w} \in W_b$ s.t. $\tilde{w} = \tilde{c}^{\gamma}$, satisfying $F_I[b] \setminus \{\tilde{c}\}$. Evidently, each member of F_I containing $\tilde{c}$ has a unique distinct literal over a variable in $\hat{b}$ which has to be assigned to 1; let w be the corresponding extension of $\tilde{w}$. Thus w satisfies F_I, and there remain

$$|\hat{b}| - \tilde{m} \geq (2^{|b|} - 2)\tilde{m} + m_{c'} - 1 \geq 2^{|b|} - 2$$

unassigned variables of $\hat{b}$. It is $|F_{II}| = |b| + 1$ and we either have $|b| \geq 3$ implying

$$2^{|b|} - 2 > |b| + 1$$

so F_{II} is satisfiable, because each of its clauses can be solved by an independent variable. Or $|b| = 2$, meaning two unassigned variables which both occur in each of three clauses. So there is a variable occurring in two of the clauses as the same literal, hence it can be set accordingly to solve them. The third clause can be solved by the second variable, so also F_{II} is satisfiable meaning $F \in \mathrm{SAT}$.

Case (iii): $F_I[b] = W_b$, and $F_{II}[\hat{b}] = W_{\hat{b}}$, then

$$|b| \geq 2^{|\hat{b}|} - 1, \quad |\hat{b}| \geq 2^{|b|} - 1$$

implying

$$\log_2(|b| + 1) \geq |\hat{b}| \geq 2^{|b|} - 1$$

which evidently is valid iff $|b| = 1 = |\hat{b}|$, contradicting the assumption and excluding this case. Hence (1) is true.

Since $\beta(\mathcal{H}_1) = \beta(\mathcal{H}_2) = 0$, one has $\mathcal{H} = \bigcup_{j\in[m]} \mathcal{H}(j)$ as a vertex-disjoint union, which has to be connected if $\mathcal{H}$ is minimal diagonal, so $m = 1$. Assertion (1) implies $|b_1| = |\hat{b}_1| = 1$. Conversely, if either of $\mathcal{H}_1, \mathcal{H}_2$ consists of exactly one loop, then $\mathcal{H}$ is isomorphic to $\mathcal{K}_2 \in \mathfrak{H}_{\mathrm{mdiag}}$ yielding (2).

According to (1) and its proof, $\mathcal{H} \in \mathfrak{H}_0$ is equivalent with $\mathcal{H} = \bigcup_{j\in[m]} \mathcal{H}(j) =: (V, B)$ as vertex-disjoint union, and $\mathcal{H}(j) =: (V_j, B_j)$ is loopless, for all $j \in [m]$. So, let $j \in [m]$ s.t. $\mathcal{H}(j)$ is maximal non-diagonal. Then there is $b'_j \subseteq V_j \subseteq V$, $b'_j \notin B_j$. By the vertex-disjointness, also $b'_j \notin B$ and $\mathcal{H}(j) \cup \{b'_j\} \in \mathfrak{H}_{\mathrm{diag}}$ implying $\mathcal{H} \cup \{b'_j\} \in \mathfrak{H}_{\mathrm{diag}}$. Hence $\mathcal{H}$ is maximal non-diagonal.

Conversely, assume that $\mathcal{H}$ is maximal non-diagonal, but $\mathcal{H}(j)$ fails to have this property, for all $j \in [m]$. Then there are $b' \subseteq V$, $b' \notin B$, s.t. $\mathcal{H}' := \mathcal{H} \cup \{b'\} \in \mathfrak{H}_{\mathrm{diag}}$, and $F' \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$ with $c' \in F'$ s.t. $V(c') = b'$. One has $F := F' \setminus \{c\} \in \mathcal{F}(\mathcal{H})$, and its restriction to B_j yields $F_j \in \mathcal{F}(\mathcal{H}(j))$. Similarly, let $c'_j := c'[V_j]$ be the restriction of c' to V_j, $j \in [m]$. Since no $\mathcal{H}(j)$ is maximal non-diagonal, it is $F_j \cup \{c'_j\} \in \mathrm{SAT}$. Finally, by the vertex-disjointness one also has

$$\begin{aligned} F' &= F \cup \{c\} = \bigcup_{j\in[m]} F_j \cup \bigcup_{j\in[m]} c'_j \\ &= \bigcup_{j\in[m]} (F_j \cup \{c'_j\}) \in \mathrm{SAT} \end{aligned}$$

providing a contradiction, yielding 3(a).

Next, let $\mathcal{H}(j)$ be dense maximal non-diagonal, for all $j \in [m]$. Fix any $b' \in B(\mathcal{K}_V) \setminus B$ and set $\mathcal{H}' := \mathcal{H} \cup \{b'\}$. Then there exists $\emptyset \neq L \subseteq [m]$ s.t. $\emptyset \neq b'_j := b' \cap V_j \notin B_j$, thus

$$\mathcal{H}'(j) := \mathcal{H}(j) \cup \{b'_j\} \in \mathfrak{H}_{\mathrm{diag}}$$

for all $j \in L$. Let $F'_j \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}'(j))$ with $c'_j \in F'_j$, $V(c'_j) = b'_j$, hence $F_j := F'_j \setminus \{c'_j\} \in \mathcal{F}(\mathcal{H}(j))$, for all $j \in L$. Choosing an arbitrary $F_j \in \mathcal{F}(\mathcal{H}(j))$, for all $j \in [m] \setminus L$, one obtains $F := \bigcup_{j\in[m]} F_j \in \mathcal{F}(\mathcal{H})$ as variable-disjoint union. Since $b' = \bigcup_{j\in L} b'_j$, for $c' := \bigcup_{j\in L} c'_j$ one has $V(c') = b'$, for which is claimed that

$$F' := F \cup \{c'\} \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$$

Indeed, suppose there exists $w \in \mathcal{M}(F)$ also satisfying c', then there is $l \in c' \cap w \neq \emptyset$. Thus there is $j \in L$ s.t. $l \in c'_j$ satisfied by w, meaning $F'_j \in \mathrm{SAT}$ providing a contradiction.

Finally, let $\mathcal{H} = \bigcup_{j\in[m]} \mathcal{H}(j) \in \mathfrak{H}_0$ be dense maximal non-diagonal and suppose there exists $j \in [m]$, for which there is $b'_j \in B(\mathcal{K}_{V_j}) \setminus B_j$, so $b'_j \notin B$, s.t. $\mathcal{H}(j) \cup \{b'_j\} \in \mathfrak{H}_0$. Hence, $\mathcal{H} \cup \{b'_j\} \in \mathfrak{H}_0$ yielding (3)(b). □

Aiming at a more specific criterion than Thm. 4 3(a), the next result is helpful:

Lemma 4: Let $b, \hat{b}$ be no loops s.t. $b \cap \hat{b} = \emptyset$. Then $\mathcal{H}(b \boxtimes \hat{b}) \notin \mathfrak{H}_{\mathrm{maxnd}}$ iff $|b| > 2$, $|\hat{b}| > 2$.

PROOF. By Thm. 4 (1), it is ensured that $\mathcal{H} := \mathcal{H}(b \boxtimes \hat{b}) =: (V, B) \in \mathfrak{H}_0$. Regarding the necessity, we show that if one of $b, \hat{b}$ has size 2 then $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$. So, w.l.o.g. suppose that $b = \{x, y\}$, $|\hat{b}| \geq 2$. As $\mathcal{H}$ is loopless, it suffices to identify a loop, say $\{x\}$, s.t. $\mathcal{H} \cup \{x\}$ becomes diagonal, to conclude that $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$. For $\tilde{\mathcal{H}} := \mathcal{H} \setminus \{\hat{b}\}$, the retraction $\tilde{\mathcal{H}}[V \setminus \{x\}]$ is well defined (cf. [16]); it actually results via removing x from each edge of $\tilde{\mathcal{H}}$. Setting $b' := b \setminus \{x\}$ yielding a loop, one directly verifies

$$\tilde{\mathcal{H}}[V \setminus \{x\}] = \mathcal{H}(b' \boxtimes \hat{b}) \in \mathfrak{H}_{\mathrm{diag}}$$

where Thm. 4 (1) has been used. Let $\tilde{F}$ be a diagonal transversal over $\tilde{\mathcal{H}}[V \setminus \{x\}]$. Let F be obtained from $\tilde{F}$, by enlarging each of its clauses by the literal x, and finally adding the clause $\{\bar{x}\}$ over the additional loop. Evidently $F \in \mathcal{F}_{\mathrm{diag}}(\tilde{\mathcal{H}} \cup \{x\})$, establishing that $\tilde{\mathcal{H}} \cup \{x\} \subseteq \mathcal{H} \cup \{x\}$ both are diagonal.

Conversely, let $2 < r \leq s$ be integers, s.t.

$$b = x_1 \cdots x_r, \quad \hat{b} = y_1 \cdots y_s$$

Again $\mathcal{H}$ is composed of the edge-disjoint parts $\mathcal{H}_I$, $\mathcal{H}_{II}$, where $\mathcal{H}_I$ results from $\mathcal{H}(b \otimes \hat{b})$ when $\hat{b}$ is replaced by b, and $\mathcal{H}_{II}$ from $\mathcal{H}(\hat{b} \otimes b)$ via replacing b by $\hat{b}$. Given $F \in \mathcal{F}(\mathcal{H})$, there are

$$F_I \in \mathcal{F}(\mathcal{H}_I),\ F_{II} \in \mathcal{F}(\mathcal{H}_{II}), \quad \text{s.t.} \quad F = F_I \cup F_{II}$$

as disjoint union. In view of La. 6 in [16], we show that adding a loop to $\mathcal{H}$ remains the resulting BHG non-diagonal, hence $\mathcal{H}$ cannot be maximal non-diagonal. For the loop, it suffices to choose (1) any vertex of b, say x_r, (2) any vertex of $\hat{b}$, say y_s.

Regarding (1), consider $\mathcal{H} \cup \{x_r\}$ and let $\tilde{F}$ be any of its transversals. Then we may assume that the clause of $\tilde{F}$ over $\{x_r\}$ is satisfied and removed yielding $F \in \mathcal{F}(\mathcal{H})$. Let F' be obtained from F by removing all those clauses that are satisfied via x_r, and removing the literal over x_r from each remaining clause. Let $b' := b \setminus \{x_r\}$, then by assumption

$$|b'| = r - 1 > 1, \quad |\hat{b}| = s > 1$$

So, according to the cases (i), (ii) in the proof of Thm. 4 (1), one concludes $F_I[b'], F_{II}[\hat{b}] \in \mathrm{SAT}$ implying $F_I(b')$, $F_{II}(\hat{b}) \in \mathrm{SAT}$. Since

$$F' \subseteq F_I(b') \cup F_{II}(\hat{b})$$

it is $F' \in \mathrm{SAT}$. The case (2), for y_s, proceeds analogously, because $s \geq r$. □

Corollary 2: Let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(m)$, $\beta(\mathcal{H}_i) = 0$, $i \in [2]$, be vertex-disjoint and loopless with $B_1 = \{b_j : j \in [m]\}$, $B_2 = \{\hat{b}_j : j \in [m]\}$. Then $\mathcal{H}_1 \boxtimes \mathcal{H}_2 \in \mathfrak{H}_{\mathrm{maxnd}}$ iff there is $j \in [m]$ s.t. $|b_j| = 2$ or $|\hat{b}_j| = 2$. □

It remains, to provide a more specific characterization for the dense maximal non-diagonality. Observe that if $\mathcal{H}(j) = (V_j, B_j)$, $j \in [m]$, as defined in Thm. 4, is loopless, then it never has V_j as an edge. Further, in the context of Thm. 4 (3)(b), one has the following criterion:

Lemma 5: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_0$ s.t. $V \notin B$. Then $\mathcal{H}$ is dense maximal non-diagonal iff there exists $F \in \mathcal{F}(\mathcal{H})$ with $|\mathcal{M}(F)| = 1$.

PROOF. The sufficiency directly follows from Thm. 5 (2) in [16]. Reversely, suppose that $|\mathcal{M}(F)| \geq 2$, for all $F \in \mathcal{F}(\mathcal{H})$, and set $\mathcal{H}' := \mathcal{H} \cup \{V\}$ which by assumption is distinct to $\mathcal{H}$. Let $F' \in \mathcal{F}(\mathcal{H}')$ be arbitrary with $c \in F'$, $V(c) = V$, then $F := F' \setminus \{c\} \in \mathcal{F}(\mathcal{H})$. Consider distinct $w_1, w_2 \in \mathcal{M}(F)$ then neither of them satisfies F' iff $w_1 = c^\gamma = w_2$. Thus there is a model of F' implying that $\mathcal{H}' \in \mathfrak{H}_0$, meaning that $\mathcal{H}$ fails to be dense maximal non-diagonal. □

In summary we obtain:

Theorem 5: Let $\mathcal{H}_i \in \mathfrak{H}(m)$, $\beta(\mathcal{H}_i) = 0$, $i \in [2]$, be vertex-disjoint with fixed labeled edge sets. Then $\mathcal{H}_1 \boxtimes \mathcal{H}_2$ is dense maximal non-diagonal iff $\mathcal{H}_i$ is 2-uniform, $i \in [2]$.

PROOF. Let $\mathcal{H}_i = (V_i, B_i)$, $i \in [2]$, with

$$B_1 = \{b_j : j \in [m]\}, \quad B_2 = \{\hat{b}_j : j \in [m]\}$$

Clearly, $\beta(\mathcal{H}_i) = 0$, $i \in [2]$, means

$$\mathcal{H} := \mathcal{H}_1 \boxtimes \mathcal{H}_2 = \bigcup_{j \in [m]} \mathcal{H}(j), \quad \mathcal{H}(j) := \mathcal{H}(b_j \boxtimes \hat{b}_j)$$

According to Thm. 4 (3)(b), it suffices to show that $\mathcal{H}(j)$ is dense maximal non-diagonal iff $|b_j| = |\hat{b}_j| = 2$, for any fixed $j \in [m]$.

Concerning the necessity, first let $|b_j| > 2$ and $|\hat{b}_j| > 2$, then $\mathcal{H}(j) \notin \mathfrak{H}_{\mathrm{maxnd}}$ by La. 4, thus it cannot be dense maximal non-diagonal.

If w.l.o.g. $|b_j| = 2$, $|\hat{b}_j| > 2$, it is claimed that $|\mathcal{M}(F)| \geq 2$, for an arbitrary $F \in \mathcal{F}(\mathcal{H}(j))$, yielding the assertion on basis of La. 5. To establish this claim, we refer to the notation in the proof of Thm. 4 (1), and let

$$U_{b_j} := W_{b_j} \setminus F_I[b_j], \quad U_{\hat{b}_j} := W_{\hat{b}_j} \setminus F_{II}[\hat{b}_j]$$

By assumption $|U_{\hat{b}_j}| \geq 5$. It either is $|U_{b_j}| > 0$, so choose any $c \in U_{b_j}$ and assign all its literals to 0, yielding a partial model $w \in W_{b_j}$ satisfying $F_I[b_j]$, so F_I. Independently fix distinct $\hat{c}$, $\hat{c}' \in U_{\hat{b}_j}$. Assigning all literals in $\hat{c}$, $\hat{c}'$ to 0, provides the partial models $\hat{w}, \hat{w}' \in W_{\hat{b}_j}$, respectively, each satisfying $F_{II}[\hat{b}_j]$, so also F_{II}. By enlargement about w, two distinct models for $F = F_I \cup F_{II}$ are obtained.

Or it is $|U_{b_j}| = 0$, for which the argumentation of the case (ii) in the proof of Thm. 4 (1) is slightly adapted: There is a unique $c' \in F_I \cap W_{b_j}$, and a unique $c'' \in F_I[b_j]$, each of whose literals occurs exactly once in F_{II}. Again m_c is the number of clauses in F_I containing $c \in F_I[b_j]$. Set

$$\tilde{m} := \min\{m_c : c \in F_I[b_j] \setminus \{c', c''\}\} \geq 1$$

Thus if $c' = c''$, it is

$$\begin{aligned} |\hat{b}_j| &= |F_I \setminus \{c'\}| \geq (2^{|b_j|} - 1)\tilde{m} + m_{c'} - 1 \\ &= 3\tilde{m} + m_{c'} - 1 \end{aligned}$$

And if $c' \neq c''$, it is

$$|\hat{b}_j| \geq 2\tilde{m} + m_{c'} + m_{c''} - 1$$

Fix $\tilde{c} \in F_I[b_j] \setminus \{c', c''\}$ with $m_{\tilde{c}} = \tilde{m}$, which always exists. As $F_I[b_j] \in \mathcal{I}$, there is $\tilde{w} \in W_{b_j}$ s.t. $\tilde{w} = \tilde{c}^\gamma$, satisfying $F_I[b_j] \setminus \{\tilde{c}\}$. Each clause of F_I containing $\tilde{c}$ has a unique literal over a variable in $\hat{b}_j$, which is forced to be assigned to 1 yielding the extension w of $\tilde{w}$. Hence, w satisfies F_I and at least one clause of F_{II}. There remain

$$|\hat{b}_j| - \tilde{m} \geq 2\tilde{m} + m_{c'} - 1 \geq 2$$

unassigned variables of $\hat{b}_j$, if $c' = c''$. And if $c' \neq c''$, there remain

$$\tilde{m} + m_{c'} + m_{c''} - 1 \geq 2$$

unassigned variables of $\hat{b}_j$. It is $|F_{II}| = |b_j| + 1 = 3$ of which at least one clause is solved via c''. Thus, there remain at most 2 unsolved clauses, and both variables occur in these clauses. If there is a pure literal for one variable, two models are obtained by assigning both values to the other variable. If there is no pure literal, assigning the first variable accordingly for the first clause, forces the other variable in the second clause. As there is no pure literal, assigning the first variable complementary, solves the second clause and forces the other variable in the first clause, yielding two models for F.

Regarding the sufficiency, for arbitrary fixed $j \in [m]$, set

$$b_j =: b = x_1 x_2, \quad \hat{b}_j =: \hat{b} = y_1 y_2$$

Then due to Thm. 4 (1), it is $\mathcal{H}(j) \in \mathfrak{H}_0$. Further,

$$F_j := \{x_1 x_2 y_1, \bar{x}_1 \bar{x}_2 y_2, \bar{x}_1 x_2, \bar{y}_1 y_2 x_1, y_1 \bar{y}_2 \bar{x}_2, \bar{y}_1 \bar{y}_2\}$$

lies in $\mathcal{F}(\mathcal{H}(j))$. Finally, one has $|\mathcal{M}(F_j)| = 1$, from which the desired assertion and the theorem follow using La. 5. Indeed, $x_1 = y_1 = y_2 = 0$, $x_2 = 1$, provides a model of F_j, which can easily be shown to be solely. □

It is not hard to verify, that the previous results, concerning the symmetric join of trivial BHGs, are valid also for their symmetric τ-join:

Corollary 3: Let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}(m)$, $\beta(\mathcal{H}_i) = 0$, $i \in [2]$, be vertex-disjoint, loopless. Then Thm. 4, Cor. 2, and Thm. 5 remain true if $\mathcal{H}_1 \boxtimes_{\mathrm{id}} \mathcal{H}_2$ is substituted by $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$, for an arbitrary $\tau \in S_m$. □

In case that $\mathcal{H}_1, \mathcal{H}_2$ are non-trivial, non-diagonal and 2-uniform, $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$ needs not to be non-diagonal. Adapting Thm. 1 (8) using certain assumptions on $\mathcal{H}_1, \mathcal{H}_2$, one is enabled to derive conditions for the non-diagonality of $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$. Moreover, if the vertex-disjoint $\mathcal{H}_1, \mathcal{H}_2$ both are dense maximal non-diagonal, one verifies, analogously to the argumentation for Thm. 4 3(b), that the same is true for $\mathcal{H}_1 \cup \mathcal{H}_2 \subsetneq \mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$ implying its diagonality. Slightly more general, one has a "local" criterion for non-diagonality without explicit further assumptions for $\mathcal{H}_1, \mathcal{H}_2$:

Theorem 6: Let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}_0(m)$, $i \in [2]$, $V_1 \cap V_2 = \emptyset$, $\tau \in S_m$, and $\mathcal{H}_\tau := \mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$. If there exists a dense maximal non-diagonal $\mathcal{H}_d = (V_d, B_d) \subseteq \mathcal{H}_\tau$ with $|V_d \cap V_i| \geq 2$, $i \in [2]$, s.t. there is $b \in B_1$, $\hat{b} := \tilde{\tau}(b) \in B_2$ with

$$(b \cap V_d) \cup (\hat{b} \cap V_d) \neq \emptyset \quad \textit{and} \quad |b \setminus V_d| \leq 2,\ |\hat{b} \setminus V_d| \leq 2$$

then $\mathcal{H}_\tau \in \mathfrak{H}_{\mathrm{diag}}$.

PROOF. Since $|V_d \cap V_i| \geq 2$, $i \in [2]$, it is $V_d \notin B_d$. Hence, there is $F_d \in \mathcal{F}(\mathcal{H}_d)$ with $|\mathcal{M}(F_d)| = 1$ by La. 5 and the assumption. Let w_d be this model, and $\mathcal{H}(b \boxtimes \hat{b}) =: \mathcal{H}_{\mathrm{loc}}$. Partially define $F \in \mathcal{F}(\mathcal{H}_{\mathrm{loc}})$ via setting x according to $w_d^\gamma(x)$ in all its occurrences of $B(\mathcal{H}_{\mathrm{loc}})$, for all $x \in V_d$. Meaning that all these literals are assigned to 0 and are removed. This in particular yields the reduced edges

$$b' := b \setminus V_d,\ \hat{b}' := \hat{b} \setminus V_d, \ \text{ so } \ 0 \leq |b'| \leq 2,\ 0 \leq |\hat{b}'| \leq 2$$

If $|b'| = 0$ or $|\hat{b}'| = 0$, we are done. If there is a loop among b', $\hat{b}'$, we are done also, because according to La. 3 it is $\mathcal{H}(b' \boxtimes \hat{b}') \in \mathfrak{H}_{\mathrm{diag}}$.

Otherwise, both b', $\hat{b}'$ are 2-uniform by assumption, and the remaining part of F over $B(b' \boxtimes \hat{b}')$, denoted as F', can be constructed to yield an unsatisfiable transversal over $\mathcal{H}_{\mathrm{loc}}$ as follows: Due to the end of the proof of Thm. 5, there exists a transversal over $\mathcal{H}(b' \boxtimes \hat{b}')$, which is chosen for F', having a unique model w'.

Moreover, we may assume w.l.o.g. that $\hat{b} \cap V_d \neq \emptyset$, meaning $|\hat{b}| \geq 3$. So, there is an additional reduced 2-clause c' over b' which is not in F'. Setting all literals in c' according to ${w'}^\gamma$, provides $F' \cup \{c'\} \in \mathrm{UNSAT}$. If c denotes the enlargement of c' about the literals over V_d as defined above, one obtains $F \cup \{c\}$, which is a diagonal transversal over (a sub-hypergraph of) $\mathcal{H}_{\mathrm{loc}} \subseteq \mathcal{H}_\tau$. $\square$

A concrete example, for $\mathcal{H}_d$ as defined in the previous (local) criterion, is provided by the 2-uniform BHG defined at the end of the proof of Thm. 5. Further, since the proof above relies on the special structure of $\mathcal{H}(b \boxtimes \hat{b})$, the criterion cannot be adapted to $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2$.

VII. Conclusions and Open Problems

Whereas several conditions are provided in Thms. 1 and 2, a complete characterization of the necessary and sufficient properties of $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}(m)$ s.t. $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2 \in \mathfrak{H}_0$, still is open. The assertions of Thm. 2 are conjectured to remain true, for every $\mathcal{H}_1$ with $\beta(\mathcal{H}_1) = 0$.

For a fixed $\tau \in S_m$, setting

$$\mathcal{F}(\mathcal{H}_1) \otimes_\tau \mathcal{F}(\mathcal{H}_2) := \{F_1 \otimes_\tau F_2 : (F_1, F_2) \in \mathcal{F}(\mathcal{H}_1) \times \mathcal{F}(\mathcal{H}_2)\}$$

one clearly has

$$\mathcal{F}(\mathcal{H}_1) \otimes_\tau \mathcal{F}(\mathcal{H}_2) \subsetneq \mathcal{F}(\mathcal{H}_1 \otimes_\tau \mathcal{H}_2)$$

even in case of trivial $\mathcal{H}_1, \mathcal{H}_2$. However, in the diagonal case it would be challenging to completely characterize the conditions for $\mathcal{H}_1, \mathcal{H}_2, \tau$, s.t.

$$\mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1) \otimes_\tau \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_2) = \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_1 \otimes_\tau \mathcal{H}_2)$$

Further, one should state precise conditions for $\mathcal{H}_1, \mathcal{H}_2, \tau$, ensuring that $\mathcal{H}_1 \otimes_\tau \mathcal{H}_2$ is (dense) maximal non-diagonal, respectively minimal diagonal, and vice versa, cf. Prop. 1, Prop. 2. Refering to Prop. 6 (2), it was desirable to become a thorough knowledge regarding the parameter φ_p. A complete clarification concerning the τ-dependence of $\delta(\mathcal{H}_1 \otimes_\tau \mathcal{H}_2)$ in the diagonal case fails to exist, cf. Prop. 7.

The structure of $\{C_1 \otimes_\sigma C_2 : \sigma \in S_m\}$, for $C_i \in \mathrm{CNF}(V_i, m)$, $i \in [2]$, can be investigated independently in the case, that the σ-join fails to be fibre-respecting which especially includes the case $|\mathcal{H}(C_1)| \neq |\mathcal{H}(C_2)|$. Otherwise, the analysis of

$$\{\mathcal{H}(C_1) \otimes_\tau \mathcal{H}(C_2) : \tau \in S_{m'}\}$$

for $\mathcal{H}(C_i) \in \mathfrak{H}(V_i, m')$, $i \in [2]$, is required also.

Refering to Thm. 6, the general characterization regarding the (non-)diagonality of $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$ is open, which also holds true for its minimal diagonality, respectively its (dense) maximal non-diagonality. The same lack is given concerning the τ-dependence of the orbit parameters of $\mathcal{H}_1 \boxtimes_\tau \mathcal{H}_2$.

Finally, clarifying the general structure of the symmetric σ-join of formulas together with its fibre-respecting specialization yields a project for future work.

References

[1] R. Aharoni, N. Linial, "Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas," *J. Comb. Theory, Series A* vol. 43, pp196-204, 1986.

[2] B. Aspvall, M. R. Plass and R. E. Tarjan, "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Inform.. Process. Lett.*, vol. 8, pp121-123, 1979.

[3] C. Berge, *Hypergraphs*, North-Holland, Amsterdam, 1989.

[4] E. Boros, Y. Crama and P. L. Hammer, "Polynomial-time inference of all valid implications for Horn and related formulae," *Ann. Math.. Artif. Intellig.*, vol. 1, pp21-32, 1990.

[5] E. Boros, Y. Crama, P. L. Hammer and X. Sun, "Recognition of q-Horn formulae in linear time," *Discrete Appl. Math.*, vol. 55, pp1-13, 1994.

[6] S. A. Cook, "The Complexity of Theorem Proving Procedures," in *3rd ACM Symposium on Theory of Computing 1971*, pp151-158.

[7] R. M. Karp, "Reducibility Among Combinatorial Problems," in: *Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972*, Plenum, New York, pp85-103.

[8] D. E. Knuth, "Nested satisfiability," *Acta Informatica*, vol. 28, pp1-6, 1990.

[9] J. Kratochvil and M. Krivanek "Satisfiability of co-nested formulas," *Acta Informatica*, vol. 30, pp397-403, 1993.

[10] H. R. Lewis, "Renaming a Set of Clauses as a Horn Set," *J. ACM*, vol. 25, pp134-135, 1978.

[11] M. Minoux, "LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation," *Inform. Process. Lett.*, vol. 29, pp1-12, 1988.

[12] S. Porschen, "A CNF Formula Hierarchy over the Hypercube," in *Lecture Notes in Artificial Intelligence: Proc. AI 2007*, pp234-243.

[13] S. Porschen, "Base Hypergraphs and Orbits of CNF Formulas," *Lecture Notes in Engineering and Computer Science: Proceedings of The International Multiconference of Engineers and Computer Scientists 2018*, 14-16 March, 2018, Hong Kong, pp106-111.

[14] S. Porschen, "A Hierarchy of Diagonal Base Hypergraphs," *Lecture Notes in Engineering and Computer Science: Proceedings of The International Multiconference of Engineers and Computer Scientists 2019*, 13-15 March, 2019, Hong Kong, pp106-111.

[15] S. Porschen, "CNF-Structure: Stabilizers, Orbits, Fibre-Transversals," *Int. J. Comp. Sci.* vol. 47(2), pp284-295, 2020.

[16] S. Porschen, "CNF-Base Hypergraphs: (Dense) Maximal Non-Diagonality and Combinatorial Designs," *Int. J. Comp. Sci.* vol. 51(3), pp329-344, 2024.

[17] S. Porschen, E. Speckenmeyer and X. Zhao, "Linear CNF formulas and satisfiability," *Discrete Appl. Math.* vol. 157, pp1046-1068, 2009.

[18] C. A. Tovey, "A Simplified NP-Complete Satisfiability Problem," *Discrete Appl. Math.* vol. 8, pp85-89, 1984.

[19] J. H. van Lint, and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 2001.