Global Output Convergence of RNN in Menger Probabilistic Metric Space

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***Abstract*— This paper discusses the global output convergence for continuous time recurrent neural networks with continuous decreasing as well as increasing activation functions in probabilistic metric space. We establish three sufficient conditions to guarantee the global output convergence of this class of neural networks. The present result does not require symmetry in the connection weight matrix. The convergence result is useful in the design of recurrent neural networks with different converging conditions.**

***Index Terms*— Global output convergence, neural networks, probabilistic metric space.**

I. Introduction

Generally speaking, a memory is a system with three functions or stages: 1) Recording: storing the information; 2) Preservation: keeping the information safely; 3) Recall: retrieving the information [1]. Research in psychology has shown that the human brain recalls by association, that is, the brain associates the recalled item with a piece of information or with another item [2].

In this paper, we consider a continuous-time recurrent neural networks (RNNs) given by

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} w_{ij}\phi_j(x_i(t)) + u(t)$$

$$x_i(0) = x_{i0}, \quad i = 1, 2, ...n$$

or, equivalently, in matrix format given by

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$$\frac{dx}{dt} = W\phi(x(t)) + u(t), \quad x(0) = x_0 \tag{1}$$

where $x = (x_1, x_2, ..., x_n)^T \in R^n$ is the state vector, $W = [w_{ij}] \in R^{n\times n}$ is a constant connection weight matrix, $u(t) = [u_1(t), u_2(t), ...u_n(t)]^T \in R^n$ is a nonconstant input vector function defined on$[0,+\infty)$ which is called the time varying threshold, $\phi(x) = [\phi_1(x_1), \phi_2(x_2), ...\phi_n(x_n)]^T$ is a nonlinear vector valued activation function from R^n *to* R^n , and $y = \phi(x)$ is called the output of the network(1). When $u(t)$ is a constant vector threshold, the RNN model (1) has been applied to content-addressable memory (CAM) problems in [3] and [4], and is also a subject of study in [5] and [6]. Recently, the RNN model (1) has been widely applied in solving various optimization problems such as linear programming problem [7], [8], shortest path problem [9], sorting problem [10].

The RNN model (1) is different from the well-known Hopfield neural networks which have been used in some optimization problems, e.g. [11], and [12]. In some applications of neural networks (e.g., CAM), the convergence of the network in the state space is a basic requirement [13], while in other applications (e.g., some optimization problems), only the convergence in the output space may be required [14],[15] and [16]. Recently, global asymptotic stability and global exponential stability of the Hopfield neural networks have received attention, e.g., [17], [18], [19], and [20]. Within the the class of sigmoidal activation functions, it was proved that negative semi definiteness of the symmetric connection weight matrix of a neural network model is necessary and sufficient for absolute stability of the Hopfield neural networks [21]. This paper investigate the global output convergence of the RNN model (1) by employing different converging conditions for continuous and monotone non-decreasing activation functions. Many authors have been studied the convergence theorem for obtaining the stable points in probabilistic metric space [22], [23], [24]. We establish three sufficient conditions for the global output convergence by extending the results of these papers. As a consequence, the present results expand the application domain of the RNN model (1).

The remainder of this paper is organized as follows. In Section II, some preliminaries on recurrent neural network probabilistic metric space and Menger probabilistic metric space are presented. Convergence results are developed in Section III. An illustrative example is given in Section IV. Finally we concluded the results in Section V.

II. Preliminaries and Assumptios

We assume that the function ϕ (.) in (1) belongs to the class of Menger probabilistic metric space and is continuous, monotone non decreasing distributive activation function;

that is , for ϕ(.) there exist a mapping $f : R \to R^+$ and $\inf f(x) = 0$, and $\sup f(x) = 1$.

Definition2.1: A probabilistic metric space (PM space) is an ordered pair (X, F) , X is a nonempty set and $F : X \times X \to L$, where L is the set of all distribution function, is a mapping such that (1) $F(p,q) = 1, \forall x > 0$, *iff* $p = q$ (2) $F(p,q,0) = 0$, (3) $F(p,q) = F(q,P)$ (4) $F(p,q,x) = 1$, $F(q,r,y) = 1 \Rightarrow F(p,r,(x+y)) = 1$. It can be noted that $F(p,q,x)$ is value of the function $F(p,q) \in L$ *at* $x \in R$.

Definition2.2: A mapping $t : [0,1]\times[0,1] \to [0,1]$ is called *t*-norm if it is non decreasing, commutative, associative and

$t(a,1) = a \forall a \in [0,1]$.

Definition2.3: A Menger PM space is a triple $(X, F; t)$ where (X, F) is a PM space and t is *t*-norm such that
$F(p,q,(x+y)) \geq t[F(p,q,x), F(q,r,y)]\ \forall (x,y) \geq 0$.

Definition2.4: A sequence $\{p_n\}$ in X is said to converges $p \in X$ *iff* $\forall \varepsilon > 0$ *and* $\lambda > 0$, $\exists$ an integer M such that $F(p_n, p, \varepsilon) > 1-\lambda$, $\forall n \geq M$. Again $\{p_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$ and $\lambda > 0$ $\exists$ an integer M such that $F(p_n, p_m, \varepsilon) > 1-\lambda$ $\forall m.n \geq M$.

Lemma2.5: Suppose $\{p_n\}$ is a sequence in Menger space $(X, F; t)$, where t is continuous and $t(x,x) \geq x$, $\forall x \in [0,1]$. If $\exists k \in [0,1]$ such that $\forall x > 0$ and positive integer *n* such that $F(p_n, p_{n+1}, kx) \geq F(p_{n-1}, p_n, x)$,x), then $\{p_n\}$ is a Cauchy sequence.

Remark: The above lemma can also be written as "Suppose $\{p_n\}$ is a sequence in Menger space $(X, F; t)$, where t is continuous and $t(x,x) \geq x$ $\forall x \in [0,1]$. *If* $\exists k > 1$ such that $\forall x > 0$ and positive integer n, $F(p_{n-1}, p_n, kx) \leq F(p_n, p_{n-1}, x)$, then $\{p_n\}$ is a Cauchy sequence". This is possible because if for $k > 1$ and $F(p_{n-1}, p_n, kx) \leq F_{p_n, p_{n-1}}(x)$, then $F(p_{n-1}, p_n, x)$ $\leq F(p_n, p_{n-1}, (x/k)) = F(p_n, p_{n-1}, k'(x)) \Rightarrow$ $F(p_n, p_{n-1}, k'(x)) \geq F(p_{n-1}, p_n, x)$ *where* $k' = (1/x) \in (0,1)$ so by the above lemma $\{p_n\}$ is a Cauchy sequence.

Definition2.6: Let (X, F) be a PM space and $f : X \to X$ be a mapping defined on *X*. Then f is said to converging if $\exists k \in [0,1]$ such that $\forall p, q \in X$, $F(f(p), f(q), kx) \geq\ \forall F(p,q,,x), X > 0$.

Theorem2.7: Every converging mapping has at most one stable point if exists.

Lemma2.8: If (X, d) is a metric space, then the metric d induces a mapping $F : X \times X \to X$ defined by $F(p,q,,x) = H(x - d(p,q)),\ p,q \in X$ *and* $x \in R$.

III. Convergence Analysis

In this section, we will establish the global output convergence of the RNN model (1). We first prove the following theorem that pertains to the existence and uniqueness of solution of the RNN model (1).

Theorem3.1: Let $(X, F; t)$ be a complete Menger space and $t(x,x) \geq x$ for all *x* in [0,1]. If *f* be a self mapping on *X* and $\{p_n\}$ is a Cauchy sequence defined by $\{p_n\}$ = fp_{n-1} converges to p in *X*. then *p* is a stable point of *f*.

Proof: Since p_n converges to p so $n \to \infty$ $F(p_n, p\ , x/2) = 1$. Due to continuity of f , $F(fp_n, fp\ , x/2) = 1$,again,

$F(fp, p\ , x) \geq t[F(fp\ , fp_n, x/2), F(fp_n, p, x/2)]$,

for $n \to \infty$, $F(fp, p\ , x) \geq t[1,1] = 1$ for all $x > 0$.

Therefore by the property of distribution function $f(p) = p$.

Theorem3.2: Let $(X, F; t)$ be a complete Menger space and $t(x,x) \geq x$ for all *x* in [0,1]. If f and g are two self mapping on *X*, such that

(I) $F(fp\ , fq, x) \geq F(gp, gq, x)$, for all *p, q* in *X* and *x* > 0,

(II) *f* is continuous,

(III) g is converging.

Then *f has* a unique stable point.

Proof: Suppose p_0 be an arbitrary point in *X,* construct a sequence $\{p_n\}$ defined by $p_n = fp_{n-1}$. Since g is converging so there exist a real number k in (0, 1) such that

$$F(p_n, p_{n-1}, kx) = F(fp_{n-1}, fp_n, kx)$$
$$\geq F(gp_{n-1}, gp_n, kx)$$
$$\geq F(p_{n-1}, p_n, x)$$

By lemma (2.5) $\{p_n\}$ is a Cauchy sequence. Since $(X, F; t)$ be a complete Menger space so $p_n \to p$ in *X*. Therefore by theorem (3.1), p is unique common stable point of *f*. Uniqueness follows from theorem (3.1).

Definition3.3: Let $(X, F; t)$ be a Menger space. A mapping *f,* defined on *X* is called dual convergence if there exist $k > 1$ such that, $F(fp, fqx, kx) \leq F(p, q, x)$, $x > 0$.

Theorem3.4: Let $(X, F; t)$ be a complete Menger space and $t(x, x) \geq x$ for all x in [0,1]. If f be onto self mapping on X and f is dual convergence. Then f has a unique common stable point.

Proof: If $p \neq q$ and $fp = fq$ then $1 \leq F(p, q, x)$ which is not possible because $1 < F(p, q, x)$. So *f* is one to one and onto mapping. Let $f^{-1} = g$. Then by dual convergence

$F(p, q, kx) \leq F(gp, gq, x)$,for all *p, q* in *X* and $x > 0$,

$$F(gp, gq, (1/k)x) = F(gp, gq, k'x)$$
$$\geq F(p, q, x), \ (1/k) = k' \ , \ 0 < k' < 1.$$

Then g is converging and satisfy all the conditions of theorem (3.2), so there exist p_0 in X such that, $g(p_0) = p_0 . f^{-1}(p_0) = p_0$ which implies that $f(p_0) = p_0$. Therefore, p_0 is a unique fixed point of f.

Theorem3.5: Let (X,F,t) be a complete Menger probabilistic metric space where $F_{p,q}$ is strictly increasing distribution function and $f : X \to X$ is continuous mapping. If there exist a real number k in (0, 1) such that $F(fp, fq, kx) \geq \min \{ F(p, q, x) , F(p, fp, x) , F(q, fq, x) , F(q, fp, x) , F(fq, f^2 p, x) \}$.Then, there exist a unique stable point.

Proof: Let $p_0 \in X$, Construct a sequence $pn = f(p_{n-1})$, $n = 1, 2, 3$……………Then

$$F(p_n, p_{n-1}, kx) = F(fp_{n-1}, fp_n, kx)$$
$$\geq \min\{F(p_{n-1}, p_n, x), F(p_n, p_{n-1}, x), F(p_n, p_n, x), F(p_{n-1}, p_{n-1}, x)$$

i.e $F(p_n, p_{n-1}, kx) \geq \min \{ F(p_{n-1}, p_n, x), F(p_n, p_{n-1}, x) \}$

$F(p_n, p_{n-1}, kx) \geq F(p_{n-1}, p_n, x)$, $x > 0$.

Therefore, by lemma 2.5 $\{p_n\}$ is a Cauchy sequence. Since $(X, F; t)$ is complete so $p_n \to p \in X$. Then by theorem (3.1), p is a unique stable point of *f.* For uniqueness suppose $f(p) = p$, $f(q) = q$. Then

$$F(p, q, kx) = F(fp, gq, x) \geq \min \{ F(p, q, x), F(p, p, x), F(q, q, x), F(q, p, x) \}$$

i.e. $F(p, q, kx) \geq F(p, q, x)$.

Which is not possible so $p = q$. Because $F_{p,q}$ is strictly increasing function and $kx < 0$.

Theorem3.6: Let $(X, F; t)$ be a complete Menger probabilistic metric space where $F_{p,q}$ is strictly increasing distribution function and $f, g : X \to X$ is continuous mapping. If there exist a real number k in (0, 1), such that

$F(fp, gq, kx) \geq \min \{ F(p, q, x) , F(p, fp, x) , F(q, gq, x) \}$.

Then, *f* and *g* have a unique common stable point.

Proof: Let $p_0 \in X$, Construct a sequence $\{p_n\}$ defined by $f(p_{2n}) = p_{2n+1}$, $gp_{2n+1} = p_{2n+1}$, $n = 1, 2, 3$……If $n = 2r + 1$.

then $F(p_n, p_{n+1}, kx) = \min\{F(p_{n-1}, p_n, x), F(p_n, p_{n+1}, x)\}$.

$F(p_n, p_{n+1}, kx) \geq F(p_{n-1}, p_n, x) f_{p,q} kx < x n = 2r$

$F(p_n, p_{n+1}, kx) \geq F(p_{n-1}, p_n, x)$. Because $f_{p,q}$ is strictly increasing function and $kx < x$. Again if $n = 2r$ then - $F(p_n, p_{n+1}, kx) = F(p_r, p_{2r+1}, kx) = F(gp_{2r-1}, fp_{2r}, kx)$

$\geq \min \{ F(p_{2r}, p_{2r+1}, x), F(p_{2r}, p_{2r+1}, x), F(p_{2r-1}, p_{2r}, x)\}$

$F(p_n, p_{n+1}, kx) \geq \min\{F(p_{2r}, p_{2r-1}, x), F(p_{2r}, p_{2r+1}, x)\}$

$F(p_n, p_{n+1}, kx) \geq F(p_n, p_{n-1}, x)$, $x > 0$.

Therefore for every +ve integer n

$$F(p_n, p_{n+1}, kx) \geq F(p_n, p_{n-1}, x).$$

Therefore, by lemma 2.5, $\{p_n\}$ is a Cauchy sequence. Then $p_n \to p \, \varepsilon \, X$. Since $\{p_{2n+1}\}$, $\{p_{2n}\}$ is a subsequence of $\{p_n\}$ so

$p_{2n+1} \to p$, $p_{2n} \to p$, then

$F(p) = p$ and $g(p) = p$, that is p is a common stable point of *f* and *g*. For uniqueness suppose *p* and *q* are two common stable point *f* and *g*. Then

$$F(p, q, kx) = F(fp, gq, x)$$
$$\geq \min \{ F(p, q, x) , F(p, p, x) , F(q, q, kx)\}.$$

It implies that

$F(p, q, kx) \geq F(q, p, x)$.

Which is not possible because $f_{p,q}$ is strictly increasing function and $kx < x$. Therefore f and g have a unique common fixed point.

IV. ILLUSTRATIVE EXAMPLE

In order to prove the above theorem, we can take $X = [0,1]$, a Menger space with usual metrice. Let $f,g:X\to X$ is defined as

$f(p) = 2p^3$

$g(p) = p^3$, *then*

$f(p) \geq g(p)\ \ \forall p$, *i.e.* $F(fp, fq,x) \geq F(gp, gq,x)$, $\forall\ p,q \in X$, $x > 0$. Clearly f and g satisfy all the conditions of above theorems and f has a unique stable point 0.

To verify the Theorem 3.6, let us take the example $X = [0,\infty]$ is a Menger space with usual metrice. Suppose $f,g:X\to X$ is defined as –

$f(p) = p$, *if* $p \in [0,1]$

$= 1$, *if* $p \in [1,\infty]$, and

$g(x) = p / (1+p)$, for all $p \in X$.

Then f and g will satisfy all the conditions of theorem 3.3. Further, f and g have unique common stable point 0. Since Theorem 3.2 is a particular case of Theorem 3.3, so if we take g = I (Identity map) in the above example, then, we have stable point of f .

V. CONCLUSION

In this paper, we have established global output convergence for a recurrent neural network with continuous and monotone non-decreasing activation functions in Menger probabilistic metric space. Three sufficient conditions to guarantee the global output convergence of this class of neural networks have been established. These results extend existing results and are very useful in the design of recurrent neural networks

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