Stochastic Stability of Linear Gyroscopic Dynamic Systems

Leila Fallah Araghi, Mohammad Reza Arvan

Abstract—**in this paper, a theorem is derived for the existence of a common quadratic Lyapunov function for stability analysis of linear gyroscopic dynamic systems. A new method based on stochastic stability. In this paper we study the stochastic stability properties of linear gyroscopic dynamic systems.**

Index Terms— **quadratic Lyapunov function, gyroscopic, stochastic stability**

I. INTRODUCTION

One of the most interesting phenomena for linear gyroscopic dynamic systems is that gyroscopic forces may stabilize a conservative system which would have been unstable in their absence [1]. In recent years many researchers has been worked in this area such as Ranislav. M. Bulatović used negative definite stiffness matrix for the stability of linear conservative gyroscopic systems [1] and he used the positive-definiteness of a certain matrix for the stability of linear conservative gyroscopic system [2] Christian Pommer worked on Gyroscopic stabilization and indefinite damped mechanical systems[3] L.A.Burlakova worked on Gyroscopic Stabilization with the Singular Matrix of Gyroscopic Forces[4].

Substantial attention has been paid to the problem of stability and stabilization of gyroscopic systems in the monograph [5].

A brief survey of the results obtained for this problem by the method of Lyapunov functions can be found in [6].

Corresponding Author Leila Fallah Araghi, Assistant professor in, Electrical Engineering, Engineering Facility Tehran University. (E-mail: l_fallah@srbau.ac.ir)

Mohammad Reza, Arvan, Malek ashtar University of Technology, Tehran, Iran. (Email: m_arvan@ashtar.ac.ir)

II. A REVIEW ON STABILITY ANALYSIS OF SWITCHING SYSTEM [7, 8, 9, 10]

In switched linear systems, the subsystems of which are continuous-time linear time-invariant (LTI) systems

$$\dot{x} = A_i x, i = \{1,...,n\}$$

(1)

Or a collection of discrete-time LTI systems

$$x[k+1] = A_i x[k],$$
$$k \in Z^+, i \in \{1,2,...,n\}$$

(2)

Where $A_i \in R^{n\times n}$.

The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures the quadratic stability of the switched system [15, 16]. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice [7, 8]. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities (LMIs) [7, 8]. Namely, there exists a positive definite symmetric matrix $P, P \in R^{n\times n}$, such that

$$A_i^T P + PA_i \prec 0, i \in \{1,...,n\}$$

(3)

For the continuous-time case, or

$$A_i^T PA_i - P_i \prec 0$$

(4)

For the discrete-time case, hold simultaneously. However, the standard interior point methods for LMIs may become ineffective have the number of modes increases [8].

Consider a dynamical system as

$\dot{x} = A_i x, i = \{1,...,n\}$

(5) Where the matrices, A_i belonging to set { $A_1,....,A_n$ } and A_i are constant matrices in $R^{n\times n}$. This system will be referred as the switching system.

Matrices A_i are asymptotically stable if the Eigen values of each A_i matrix lies in the open left half of the complex plan. So that matrices A_i are assumed to be Hurwitz.

An important problem is to determine necessary and sufficient condition for the existence of a quadratic Lyapunov function V(x) = x^T P x, P = P^T > 0, P belong to $R^{n\times n}$, such that $\frac{dv}{dt}$ along any trajectory of the system (1) is negative definite, or alternatively that

$$A_i^T P + PA_i = -Q_i \quad (6)$$

Where Q_i positive definite and P are negative definite. The function V(x) is a common quadratic Lyapunov function (CQLF) for the switching linear time-invariant (LTI) dynamic systems,

$\sum A_i : x^{\cdot} = A_i x, i = \{1,...,n\}$

(7)

Where A_i belong to $R^{n\times n}$.

The existence of such a Lyapunov function is sufficient to guarantee the uniform asymptotic (exponential) stability of switching system (1) [9].

Theory 1: If we consider switching linear time-invariant (LTI) dynamic systems,

$\sum A_i : \dot{z} == A_i z, i = \{1,...,n\}$

(8)

The system described by Equation (7) with initial conditions $z(t_0) = z_0$ has the following response [10]:

$$z(t) = \exp(A_i(t - t_k))\left(\prod_{j=1}^{k} M(j)\right) z_0$$

(9)

And

$$M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp(A_i h(j)) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Definition #1 [10]

The equilibrium $z = 0$ of a system described by $\dot{z} = f(z,t)$ with initial condition $z(t_0) = z_0$ is almost sure (or with probability-1) asymptotically stable at large (or globally) if for any $\beta \succ 0$ and $\varepsilon \succ 0$ the solution of $\dot{z} = f(z,t)$ satisfies

$$\lim_{\delta \to 0} \left\{ P(\sup_{t \geq \partial} \|z(t, z_0, t_0)\| \succ \varepsilon \right\} = 0$$

Where $\| z_0 \| \prec \beta$.

Corollary #1[10]

The system described by Equation (2), with update times $h(j)$ that are independent identically distributed random variable with probability distribution $F(h)$ is globally almost sure (or with probability-1) asymptotically stable around the solution $z = 0$ if $T = E[\exp(\sigma(A_i)h)] \prec \infty$ and the expected value of the maximum singular value of the test matrix :

$M, E[\|M\|] = E\left[\bar{\sigma} M\right]$, is strictly less than one,

where $M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp(A_i h(j)) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

III. THE MODEL OF LINEAR GYROSCOPIC DYNAMIC SYSTEMS [1]

Systems of interest here are linear conservative gyroscopic systems described by the Equation [1].

$M\ddot{q} + \hat{G}\dot{q} + \hat{K}q = 0$

(10)

Where *M*, *G^* and *K^* are real $n \cdot n$ matrices, *q* is the *n*-vector, and M is symmetric and positive definite ($M^T = M > 0$);

$\hat{G}$ Is skew-symmetric ($\hat{G}^T = -G\hat{}$)?

$\hat{K}$ Is symmetric and negative definite ($\hat{K}^T = K < 0$).

The vector *q* represents the generalized coordinates, *M* is the mass matrix, $\hat{G}$ describes the gyroscopic forces, and $\hat{K}$ the potential forces.

It is convenient, although not necessary, to rewrite Eq. (10) in the form:

$$\ddot{x} + G\dot{x} + Kx = 0$$

(11)

Using the transformation

$$K = M^{-1/2}\hat{K}M^{1/2}$$

$$G = M^{-1/2}\hat{G}M^{1/2}$$

$$x = M^{1/2}q$$

Here the exponent ½ indicates the unique positive definite square root of the matrix *M*. Clearly

$$G^T = -G$$

$$K^T = K$$

If we consider:

$$x = x_1 , \dot{x} = x_2$$

Then we can write Eq.11 into state space Equation:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Kx_1 - Gx_2$$

(12)

Then we can write Eq.12in to state space matrix:

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -G \end{bmatrix} X$$

(13)

IV. STABILITY ANALYSIS OF LINEAR GYROSCOPIC DYNAMIC SYSTEMS

If we consider

$A_i = \begin{bmatrix} 0 & 1 \\ -K & -G \end{bmatrix}$ Using theorem 1 and Corollary #1[18] we can say that (13) is globally almost sure (or with probability-1) asymptotically stable around the solution $\dot{X} = 0$.

V. CONCLUSION

This paper proposed a new method based on stochastic stability for linear gyroscopic dynamic systems.

REFERENCES

[1] Ranislav. M. Bulatović, "On Stability Criteria for Gyroscopic Systems with Negative Definite Stiffness" The scientific journal FACTA Universities *Series*: Mechanics, Automatic Control and Robotics Vol.2, No 10, 2000 pp. 1081 – 1087

[2] R. M. Bulatović, "A Remark on the Stability of Linear Gyroscopic Systems" The scientific journal FACTA Universities *Series*: Mechanics, Automatic, Control and Robotics Vol.2, No 7, 1997 pp. 255 - 258

[3] Christian Pommer, "Gyroscopic stabilization and indefinite damped mechanical systems" GAMM–SIAM Conference on Applied Linear Algebra Düsseldorf, July 24–27, 2006:

[4] L.A.Burlakova, "On Gyroscopic Stabilization with the Singular Matrix of Gyroscopic Forces" Works on Analytical Mechanics. - Moscow, Publ. AS USSR, 2000

[5] Merkin D.R. Gyroscopic Systems. Moscow, Nauka. 1974 (in Russian)

[6] Bulatovich R. On stability of linear potential gyroscopic systems in the cases
Of maximum potential energy. - J. of Applied. Mathematics and Mechanics, (1997) Vol. 61, no. 3, 385 {389.

[7] Hai Lin, & Panos J. Antsaklis, Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results. IEEE Transaction on Automatic Control VOL., NO. , 2008.

[8] L. Gurvitz, R. Shorten, O. Mason, On stability of switched positive linear Systems. IEEE, Transaction on Automatic control, 2007.

[9] G. Zhai, X. Xu, H.Lin, D. Liu, Extended Lie Algebric Stability Analysis For Switching Systems With Continuous-Time and Discrete–Time Subsystems. Int. J. Appl. Math. Compute. Sci., 2007, Vol. 17, No. 4, 447–454.

[10] Luis A. Montestruque, Panos J. Antsaklis , "STOCHASTIC STABILITY FOR MODEL-BASED NETWORKED CONTROL SYSTEMS" IEEE, Proceedings of the American Control Conference Denver, Colorado June 4-6, 2003

[19] L. Gurvits, A. Olshevsky, "Stability testing of matrix polytopes" arXiv: math [math.OC], 2007.