Space-Precise Computation of a Singular Nonlinear Evolution Equation for the Risk Preference

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Abstract—**We propose a space-precise computational scheme to investigate nonlinear partial differential equations (PDEs) defined on infinite interval. As a model equation, we treat the singular evolution equation for the risk preference in the optimal investment problem under the random risk process, whose unknown quantity is related to the Arrow-Pratt coefficient of absolute risk aversion with respect to the optimal value function. Numerical implementation shows that our method is robust and stable.**

***Keywords:** space-precise numerical implementation, absolute risk aversion, singular nonlinear PDE*

1 Introduction

Numerical computation, by its own nature, necessary employs various approximations; the differential quotient is replaced by the difference quotient, infinite intervals are truncated into finite intervals, and so on. These approximate procedures are justified by relevant convergence argument. In this paper, on the other hand, we propose an effective numerical method, which is precise in space varaiable, to study a singular nonlinear partial differential equation (PDE) defined on infinite interval. As a model equation, we consider the PDE derived from the Hamilton-Jacobi-Bellman (HJB) equation for the value function in the optimal investment problem.

First we explain background issues of our model equation. We recall that the study of the optimal behavior in economics environment has been an intensive subject for researches. Various models have been introduced so far and much progress has been made. For instance optimal portfolio problems are discussed in [4][6] after a pioneering work of Merton [15]. Applications to insurance are considered in [3][7][8][9][17]. We also refer to the references cited in these papers. We point out that the traditional way of investigation has been based on stochastic control and a number of authors have reduced the analysis to the treatment of the HJB equation for the value function. The resulting nonlinear equations, however, are typically hard to solve; it may be not an exaggeration to say that all that we can do is to merely guess a shape of solutions and manage to arrange the parameters. Observe the statement in [2]. Of course there are weak approaches to these equations and substantial success in mathematics was made. The notion of weak solutions, however, is a little involved and does not seem to meet the wishes of practitioners. As a result the analysis of HJB equations has certainly stayed as principal difficulties to be surmounted.

In our previous paper [1][14] we introduce a singular PDE in order to deal with such HJBs. See (7) below. Although essential difficulties are equivalent to those expressed by the HJB equation, this derived PDE has rather simple looking from the viewpoint of the theory of nonlinear PDEs. In addition, the unknown quantity is related to the Arrow-Pratt coefficient of absolute risk aversion [18] with respect to the optimal value function. In this sense our introduced PDE may be interpreted as the characteristic equation for the risk structure of the model. We do not insist that our PDE would replace the HJB itself but we at least believe that the study of this PDE is interesting as well as important and therefore worth investigating further. Here we undertake numerical treatment. The basic reference of this project is our recent publication [13], to which we refer for further examples.

The organization of the paper is as follows. In §2 we make a sketch of our model, derive a PDE, and recall certain existence theorem as well as the structure of steady state solutions. §3 is devoted to exhibiting our numerical scheme. Numerical implementation is depicted in §4. We conclude with discussions in §5.

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2 Model processes

The basic model we follow is rather standard. See Browne [3] or [1][14]. It is assumed that there is only one risky stock available for investment, whose price P_t at time t is governed by the stochastic differential equation of Black-Scholes-Merton type [5][16]

$$dP_t = P_t(\mu dt + \sigma dW_t^{(1)}),$$

where μ and σ are constants and $\{W_t^{(1)}\}_{t\geq 0}$ is a standard Brownian motion. There are also a risk process and a bond, whose price at time t are denoted by Y_t and B_t, respectively, and assumed to be evolved as

$$dY_t = \alpha dt + \beta dW_t^{(2)}, \qquad dB_t = \gamma B_t dt,$$

where α and β ($\beta > 0$) are constants and $\{W_t^{(2)}\}_{t\geq 0}$ is another standard Brownian motion. The interest rate $\gamma > 0$ is supposed to be constant and $\mu > \gamma$. It is allowed these two Brownian motions to be correlated with the correlation coefficient ρ. We prescribe $0 \leq \rho^2 < 1$ in the sequel.

The company invests in the risky stock under an investment policy f, where $f = \{f_t\}_{0\leq t\leq T}$ is a suitable, admissible adapted control process. T stands for the maturity date. Let X_t^f denote the wealth of the company at time t with $X_0 = x$, whose evolution process is given by

$$\begin{aligned} dX_t^f &= f_t \frac{dP_t}{P_t} + \gamma(X_t^f - f_t)dt + dY_t \\ &= (\gamma X_t^f + f_t(\mu-\gamma) + \alpha)dt + f_t\sigma dW_t^{(1)} + \beta dW_t^{(2)}, \\ X_0 &= x. \end{aligned}$$

The generator $\mathcal{A}^f$ of this wealth process is then expressed as

$$\begin{aligned} (\mathcal{A}^f g)(x,t) := &\frac{\partial g}{\partial t} + (f(\mu-\gamma) + \gamma x + \alpha)\frac{\partial g}{\partial x} \\ &+ \frac{1}{2}(f^2\sigma^2 + \beta^2 + 2\beta\sigma\rho f)\frac{\partial^2 g}{\partial x^2}. \end{aligned}$$

Suppose that the investor wants to maximize the utility $U(x)$ from his terminal wealth. The utility function $U(x)$ is customarily assumed to satisfy $U' > 0$ and $U'' < 0$. Let

$$V(x,t) := \sup_f E[e^{-\delta(T-t)}U(X_T^f) \mid X_t^f = x], \tag{1}$$

where δ stands for the rate at which consumption and terminal wealth are discounted. We remark that in the seminal work of Browne [3] the case of $\delta \equiv 0$ is treated.

Now the Bellman principle applied to (1) implies that the equation for V, which is called the Hamilton-Jacobi-Bellman equation, becomes

$$\sup_f\{\mathcal{A}^f V(x,t)\} = -\delta V, \quad V(x,T) = U(x). \tag{2}$$

Suppose that (2) has a classical solution V with $\partial V/\partial x > 0$, $\partial^2 V/\partial x^2 < 0$. We then infer that

$$f_t^* = -\frac{\mu-\gamma}{\sigma^2}\frac{\partial V/\partial x}{\partial^2 V/\partial x^2} - \frac{\beta\rho}{\sigma}, \tag{3}$$

where $\{f_t^*\}_{0\leq t\leq T}$ denotes the optimal policy. Placing (3) back into (2) we obtain

$$\begin{aligned} &\frac{\partial V}{\partial t} + \Big(\gamma x + \alpha - \frac{\beta\rho(\mu-\gamma)}{\sigma}\Big)\frac{\partial V}{\partial x} \\ &\quad - \frac{1}{2}\Big(\frac{\mu-\gamma}{\sigma}\Big)^2\frac{(\partial V/\partial x)^2}{\partial^2 V/\partial x^2} + \frac{1}{2}\beta^2(1-\rho^2)\frac{\partial^2 V}{\partial x^2} \\ &\quad = -\delta V \qquad \text{for } 0 < t < T \\ &V(T,x) = U(x). \end{aligned} \tag{4}$$

Browne [3] shows that if $\delta = 0$ (4) possesses a solution in the case $U(x) = \lambda - (\nu/\theta)e^{-\theta x}$ with positive constants λ, ν, θ. This utility has constant absolute risk aversion parameter θ; precisely stated, $-U''(x)/U'(x) = \theta$. Here we proceed further in the analysis of (4) along the line of [1]. We remark that in [1] the case of $\gamma \equiv 0$ and $\delta \equiv 0$ is discussed.

Let $v(x,t)$ be defined by $V(x,t) = v(Ex, F(T-t))$, where

$$E := \sqrt{\frac{(\mu-\gamma)^2}{\beta^2(1-\rho^2)\sigma^2}}, \quad F := \frac{1}{2}\Big(\frac{\mu-\gamma}{\sigma}\Big)^2.$$

We further define

$$a := \frac{E}{F}\Big(\alpha - \frac{\beta\rho}{\sigma}(\mu-\gamma)\Big), \quad b := \frac{\delta}{F},$$

and write γ/F by the same γ with abuse of notation. It follows that after a calculation

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \frac{(\partial v/\partial x)^2}{\partial^2 v/\partial x^2} + (\gamma x + a)\frac{\partial v}{\partial x} + bv, \\ &v(x,0) = U(E^{-1}x). \end{aligned} \tag{5}$$

Now we additionally introduce the next quantity.

$$r(x,t) := -\frac{\partial^2 v/\partial x^2}{\partial v/\partial x} = -\frac{\partial}{\partial x}\log\Big|\frac{\partial v}{\partial x}(x,t)\Big|. \tag{6}$$

It should be noted that (6) is related to the coefficient of absolute risk aversion. A little tedious computation then leads us to the following equation.

$$\begin{aligned} &\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\Big\{\Big(1 + \frac{1}{r^2}\Big)\frac{\partial r}{\partial x} - r^2 + (\gamma x + a)r\Big\}, \\ &r = r(x,t) \quad \text{in } (x,t) \in \Omega_T := \boldsymbol{R}^+ \times (0,T) \end{aligned} \tag{7}$$

where $T > 0$ and $\boldsymbol{R}^+ = \{x > 0\}$.

We remark that in [1] the equation (7) with $\gamma \equiv 0$ and $a \equiv 0$ is derived. Compared to the equation (5), which is in itself worth investigating further, the equation (7) is quasilinear and has divergence form; (7) is rather popular type in the PDE world, although it is singular at the same time.

In [14] we have proved the next existence result, which is formulated on a bounded interval.

Theorem. *Let $L, l > 0$ be given. For every non-increasing $r_0(x) \in C^1[0, L]$ with $r_0 \geq l$ and $\partial r_0/\partial x = 0$ at $x = 0$ and L, there corresponds $T = T(u_0) > 0$ such that there exists a classical solution r of the problem*

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\Big\{\Big(1+\frac{1}{r^2}\Big)\frac{\partial r}{\partial x} - r^2 + (\gamma x + a)r\Big\}$$
$$\text{in } (x,t) \in \Omega_T^L := (0, L) \times (0, T)$$
$$\frac{\partial r}{\partial x}(0,t) = \frac{\partial r}{\partial x}(L,t) = 0 \quad \text{for } 0 < t < T$$
$$\frac{\partial r}{\partial x} \leq 0, \quad r(x,t) \geq l > 0 \quad \text{for } (x,t) \in \Omega_L^T$$
$$r(x,0) = r_0(x) \quad \text{on } 0 \leq x \leq L.$$

We note that in [1] the existence is assured under a different situation; that is, the equation with $\gamma = a = 0$ is discussed on $\{x > 0\}$ under the conditions that $\partial r/\partial x = 0$ at $x = 0$ and u converges to a positive constant as $x \to \infty$.

The structure of steady state solutions makes us realize what kind of solutions the equation (7) produces. Here we reproduce our previous establishment of [1] for the readers' convenience. That is, we examine the case $\gamma = a = 0$ and consider the next ordinary differential equation:

$$\Big(1+\frac{1}{r^2}\Big)\frac{dr}{dx} - r^2 = C,$$

where C denotes a constant independent of x and t.

There are three possibilities according to the sign of C. We note that, however, the first two cases are meaningless for the economics because r takes a negative value.

If $C > 0$ then we write $C = M^2$ to obtain

$$-\frac{1/M^2}{r(x)} + \Big(\frac{1}{M} - \frac{1}{M^3}\Big)\tan^{-1}\frac{r(x)}{M}$$
$$= x - \frac{1/M^2}{r(0)} + \Big(\frac{1}{M} - \frac{1}{M^3}\Big)\tan^{-1}\frac{r(0)}{M}.$$

It follows that $r(x) \sim -M^{-2}x^{-1}$ as $x \to \infty$.

If $C = 0$ then we know

$$-\frac{1}{r(x)} - \frac{1}{3r(x)^3} = x - \frac{1}{r(0)} - \frac{1}{3r(0)^3}.$$

It follows that $r(x) \sim -(3x)^{-1/3}$ as $x \to \infty$.

Consequently if $C \geq 0$ then there is no steady state solution suitable to the finance; the risk preference should be non-negative by definition. Only the next last case fits into our requirement.

If $C < 0$ then we write $C = -M^2$ to discover

$$\frac{1/M^2}{r(x)} + \frac{1+M^2}{2M^3}\log\Big|\frac{r(x)-M}{r(x)+M}\Big|$$
$$= x + \frac{1/M^2}{r(0)} + \frac{1+M^2}{2M^3}\log\Big|\frac{r(0)-M}{r(0)+M}\Big|,$$

provided $r(x) \neq M$. In this case $r(x) \sim M^{-2}x^{-1}$ as $x \to \infty$. It is also clear that $r(x) \equiv M$ gives one of steady state solutions, which has a character of constant absolute risk aversion. It should be noted that the last steady state solutions correspond to those presented in [3].

3 Numerical Procedure

Now we turn our attention to the numerical treatment of the equation (7) under the next condition.

$$\frac{\partial r}{\partial x}(0,t) = 0, \quad \frac{\partial r}{\partial x}(x,t) \leq 0, \qquad r(x,t) \to 0 \quad \text{as } x \to \infty. \tag{8}$$

We notice that in particular the domain is not bounded but the half line.

In order to compute the equation on a bounded interval, we make the next change of variables, which is introduced by [10] in this context and successfully applied to nonlinear PDEs of this kind [11]. See also [12].

$$s = \frac{2x}{1+\sqrt{4x^2+1}} \quad \Big(x = \frac{s}{1-s^2}\Big),$$
$$r(s,t) = r(x,t)$$

The half line $[0, \infty)$ then corresponds bijectively to $[0, 1)$. Furthermore the equation (7) is transformed into

$$\frac{\partial r}{\partial t} = \Big(1+\frac{1}{r^2}\Big)\frac{(1-s^2)^4}{(1+s^2)^2}\frac{\partial^2 r}{\partial s^2} - \frac{2(1-s^2)^4}{r^3(1+s^2)^2}\Big(\frac{\partial r}{\partial s}\Big)^2$$
$$-2\Big\{\Big(1+\frac{1}{r^2}\Big)\frac{s(1-s^2)(3+s^2)}{(1+s^2)^3}$$
$$+((r-a)(1-s^2)-\gamma s)\frac{1-s^2}{1+s^2}\Big\}\frac{\partial r}{\partial s} + \gamma r$$
$$r = r(s,t) \qquad \text{in } (s,t) \in (0,1) \times (0,T)$$
$$r(s,0) = r_0\Big(\frac{s}{1-s^2}\Big) \qquad \text{on } s \in [0,1].$$

As to the first and the third condition of (8) we simply put

$$\frac{\partial r}{\partial s}(0,t) = 0, \quad r(1,t) = 0 \qquad \text{for } t \in [0,T]. \tag{9}$$

Our discretization is now performed on this transformed equation. For integers $N, M >> 1$ we define $s_i := ih$

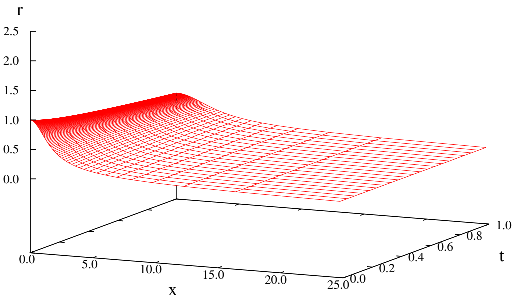
$(i = 0, 1, 2, \cdots, N)$ with $h = 1/N$, $t_j := j\Delta t$ $(j = 0, 1, 2, \cdots, M)$ with $\Delta t = T/M$, and $r_{i,j} := r(s_i, t_j)$. The boundary conditions are

$$r_{N,j} = 0, \qquad \frac{-r_{2,j} + 4r_{1,j} - 3r_{0,j}}{2h} = 0,$$

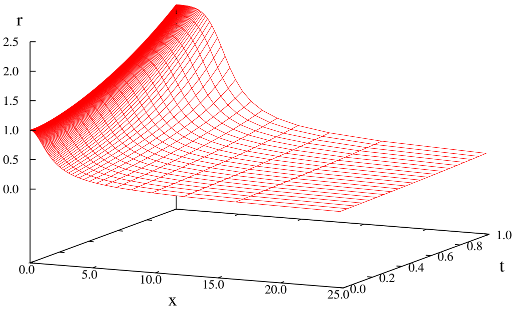
and our scheme is, for $i = 0, 1, 2, \cdots, N-1$ and $j = 0, 1, 2, \cdots, M$,

$$\begin{aligned}
&\frac{1}{\Delta t}(r_{i,j+1} - r_{i,j}) = \\
&\quad \left(1 + \frac{1}{r_{i,j}^2}\right)\frac{(1-s_i^2)^4}{(1+s_i^2)^2}\frac{r_{i+1,j} - 2r_{i,j} + r_{i-1,j}}{h^2} \\
&\quad - \frac{2(1-s_i^2)^4}{r_{i,j}^3(1+s_i^2)^2}\frac{(r_{i+1,j} - r_{i-1,j})^2}{4h^2} \\
&\quad - 2\Big\{\left(1 + \frac{1}{r_{i,j}^2}\right)\frac{s_i(1-s_i^2)^3(3+s_i^2)}{(1+s_i^2)^3} \\
&\quad + ((r_{i,j} - a)(1-s_i^2) - \gamma s_i)\frac{1-s_i^2}{1+s_i^2}\Big\}\frac{r_{i+1,j} - r_{i-1,j}}{2h} \\
&\quad + \gamma r_{i,j}.
\end{aligned} \tag{10}$$

We should remark that the condition (9), namely the point $s = 1$, does not produce a singularity.



(a) $\gamma = 0$



(b) $\gamma = 1$

Figure. Solution profiles for Example: $r_0(x) = x^{-1}\tanh x$.

4 Numerical implementation

We carry out our numerical experiment based on the scheme (10). We here consider one example. For other examples, we refer to [13]. To keep our situation simple enough we assume $a = 0$ here.

Example. We take $r_0(x) = x^{-1}\tanh x$. The results are shown in Figure. Figure (a) deals with $\gamma = 0$ while Figure (b) with $\gamma = 1$. In both cases we set $N = 10^3$, $M = 10^7$ and $T = 1$.

The function $r_0(x)$ is monotone decreasing and attains its maximum at $x = 0$. It is seen that the point of local maximum grows as time proceeds.

Other examples show that our scheme is robust and stable. We will not get into the details and just refer to [13].

5 Conclusions and Discussions

We have numerically investigated singular nonlinear partial differential equations (PDE) arising in the optimal investment problem. The unknown quantity is related to the Arrow-Pratt coefficient of absolute risk aversion in terms of the optimal value function. Therefore the current PDE may be considered to characterize the evolution of the risk preference in optimal investment behavior. It is also to be noticed that the resulting PDE is quasilinear and hence typical from the standpoint of the theory of nonlinear PDEs.

Numerical performances show that the solution is smoothened as time proceeds. Since we make a change of time inversion, this means that as time approaches the maturity it is becoming more risk sensitive. We also observe that under the effect of interest rate the solution grows. Interpretation of this phenomena in the economics is that the interest rate tempers the risk evolution. We are able to judge that our PDE really represents the risk preference in some sense.

The research for the optimal behaviors under stochastic processes gradually gains its importance both among academics and practical world. PDE approach has an advantage that it is rather easy to implement the equation by numerical computation. We hope that our methodology have made at least a positive step toward the better understanding of the optimal decision problem.

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