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Stokes Flow of an Incompressible Couple Stress Fluid past a Porous Spheroidal Shell

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***Abstract*—**The present paper deals with the problem of the Stokes flow of a couple stress fluid past a porous spheroidal shell consisting of a pair of con focal spheroids S_0 and S_1 where S_0 is within S_1. The region within S_0 is filled with couple stress fluid, the annular region between S_0 and S_1 is assumed to be porous and the same couple stress fluid as that within S_0 flows with a uniform velocity in the free region (i.e) outside S_1. The problem is formulated using the V.K. Stokes' equation describing the flow outside the shell as well as the flow inside the shell while an analogue of the classical Darcy law in the theory of porous media is used within the shell region. Under Stokesian approximation, the solution is then, sought analytically and the expressions for the flow field variables are obtained in terms of Legendre functions, associated Legendre functions, radial prolate spheroidal wave functions and angular prolate spheroidal wave functions. The stresses acting on the shell are estimated and the drag experienced by the body is obtained. Numerical study is undertaken to study the effect of the permeability of the medium, couple stress parameter and the geometric parameter on the drag and the results are presented using graphs. It is found that, for a fixed S_0 as the eccentricity of the outer spheroid increases, the drag decreases.

Index Terms— Spheroidal shell, Couple stress fluid, Drag, Streamlines.

I. INTRODUCTION

Couple stress fluid theory proposed by V.K. Stokes [1] is the simplest polar fluid theory that shows all the important features and effects of couple stresses in fluids caused due to the mechanical interactions that occur inside a deforming continuum. A striking feature of this model is that it results in equations that are similar to the Navier Stokes equations, there by facilitating a comparison with the results for the classical non polar case.

Several flows past axisymmetric bodies dealing with couple stress fluids have been studied by Lakshmana Rao, Iyengar [2] and Iyengar and Srinivasacharya [3].

All these problems deal with flows arising in the context of impervious bodies like sphere, spheroid and approximate sphere. To the extent the authors have surveyed not much literature is found on the flow of couple stress fluid past porous bodies except that by Ramana Murthy et.al [4], who studied the flow of an incompressible couple stress fluid flow past a porous sphere. Spheroid, being a more generalized form than a sphere, the present investigation deals with an incompressible couple stress fluid past a porous spheroidal shell. The studies dealing with porous spheroidal shells are of use in the study of plant physiology. This aspect has been well brought out by John Considine and Ken Brown who have used the theory of shells to study some aspects of the physics of fruit growth [5].

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As the classical Navier Stokes equations, the couple stress fluid flow equations also are non linear in nature and even of higher order than the Navier Stokes equations. Thus only very few problems can have exact solutions subject to specified boundary and regularity conditions. When we consider the problem of flow past bodies like sphere or spheroid, it is almost impossible to find an exact solution. Hence researchers tried to solve a simplified version of the fluid flow equations by imposing some assumptions based on intuition. One of these assumptions is that given by Stokes: when the flow is slow and the fluid is highly viscous, viscous forces predominate the inertial forces nearer to the body. This assumption helps in neglecting the nonlinear inertial terms in the momentum equation and there by making the problems more mathematically tractable.

In this paper, we study the Stokes flow of an incompressible couple stress fluid past a porous spheroidal shell consisting of two confocal spheroids where there is couple stress fluid filling the region inside the inner spheroid and the region outside the outer spheroid. The annular region between the two confocal spheroids is assumed to be porous in nature and the flow is governed by Darcy's law there in. The flow variables pertaining to the inner region F_0, outer region F_2 and the porous region F_1 are obtained analytically. The expressions for stream function and pressure are obtained in terms of Legendre functions, associated Legendre functions, radial prolate spheroidal wave functions and angular prolate spheroidal wave functions. The stresses acting on the spheroid are evaluated and an expression for the drag is obtained. Though the expressions for stresses are complicated, the expression for drag takes a delightfully simple form.

Numerical study is undertaken to discuss the variation of drag with respect to the material parameter, geometric parameter and the permeability parameter. The drag is seen to increase with the increase in the permeability parameter and it decreases with the increase in the eccentricity of the outer spheroid as well as an increase in the couple stress parameter.

Couple stress fluid model given by V.K. Stokes is based on the presumption that the fluent medium can sustain couple stresses. Hence we have the non symmetric stress tensor t_{ij} and the couple stress tensor m_{ij} given by

$$t_{ij} = -p\delta_{ij} + \lambda div\overline{q}\,\delta_{ij} + 2\mu d_{ij} - \frac{1}{2}\varepsilon_{ijk}\left\{m_{,k} + 4\eta_1\omega_{k,rr} + \rho C_k\right\} \quad (1)$$

$$m_{ij} = \frac{1}{3}m\delta_{ij} + 4\eta_1\omega_{j,i} + 4\eta_1'\omega_{i,j} \quad (2)$$

where $\overline{q}$ denotes the fluid velocity vector $\overline{\omega} = \frac{1}{2}curl\,\overline{q}$, $\omega_{i,j}$ is the spin tensor, d_{ij} is the rate of deformation tensor, p is the fluid pressure and ρC_k is the body couple vector. The quantities λ and μ are the viscosity coefficients and η_1, η_1' are the couple stress viscosity coefficients. These material constants are constrained by the inequalities

$$\mu \ge 0\ ,\ 3\lambda + 2\mu \ge 0\ ,\ \eta_1 \ge 0\ ,\ |\eta_1'| \le \eta_1 \quad (3)$$

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The parameter $\sqrt{\frac{\eta_1}{\mu}}$ is a characteristic measure of the polarity of the fluid which is zero in the case of non polar fluid. The couple stress fluid equations are given by

$$\frac{\partial \rho}{\partial t} + \rho div(\vec{q}) = 0 \tag{4}$$

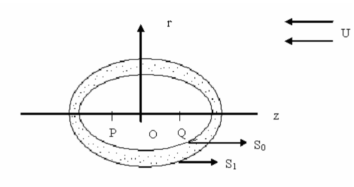
$$\rho \frac{d\vec{q}}{dt} = \rho \vec{f} + \frac{1}{2} curl(\rho \bar{c}) + div \tau^{(s)} + \frac{1}{2} curl(divM) \tag{5}$$

where ρ is the density of the fluid, $\tau^{(s)}$ is the symmetric part of the force stress diad and M is the couple stress diad and $\vec{f}$, $\bar{c}$ are the body force per unit mass and body couple per unit mass respectively

II. FORMULATION

Consider two confocal prolate spheroids S_0 and S_1 with foci P and Q where PQ=2c units. Let O be the mid point of PQ. Introduce the cylindrical polar coordinate system (r,θ,z) with respect to O as origin and OP extended on either side as Z axis.

Let us consider the slow stationary flow of an incompressible couple stress fluid past the spheroid S_1 with a uniform flow with velocity U in the direction of the z-axis far away from the body. Let the region (F_1) between S_0 and S_1 be porous. Let the region (F_0) within the spheroid S_0 be filled with the same couple stress fluid as is outside S_1.



We examine the flow generated with the assumption that the flow in the porous region is characterized by Darcy's law. Since the flow is slow, we assume that the flow is axi symmetric and is the same in any meridian plane and thus the flow variables are independent of the azimuth angle φ.

We shall introduce the prolate spheroidal coordinates (ξ,η,φ) with $(\bar{e}_\xi, \bar{e}_\eta, \bar{e}_\phi)$ as base vectors and (h_1, h_2, h_3) as the corresponding scale factors through the definition

$$z + ir = c\cosh(\xi + i\eta) \tag{6}$$

Let ($\bar{q}^i, p^i$) denote the velocity and pressure in the regions F_i (i = 0,2) and let ($\bar{q}^{(1)}, p^{(1)}$) be the velocity and pressure in the porous region F_1.
In view of the slowness of the flow, we take

$$\bar{q}^{(i)} = u^{(i)}(\xi,\eta)\bar{e}_\xi + v^{(i)}(\xi,\eta)\bar{e}_\eta \quad (i = 0,1,2) \tag{7}$$

$$p^{(i)} = p^{(i)}(\xi,\eta) \quad (i = 0,1,2) \tag{8}$$

Ignoring the body force and body couple $\bar{f}$ and $\bar{l}$ respectively in the field equations, the basic equations governing the Stokesian flow can be written in the form

$$div(\vec{q}^{(i)}) = 0 \quad \text{for i=0,1,2} \tag{9}$$

$$grad\ p^{(i)} + \mu\, \text{curl}\, curl\, \vec{q}^{(i)} + \eta_1 curl\, curl\, curl\, curl\, \vec{q}^{(i)} = 0$$
for i = 0,2 (10)
and

$$\bar{q}^{(1)} = -\frac{k^{(1)}}{\mu} gradp^{(1)} \tag{11}$$

where $k^{(1)}$ is the permeability constant.
In view of the continuity equations, we introduce the stream function $\psi^{(i)}$ through

$$h_2 h_3 u^{(i)} = -\frac{\partial \psi^{(i)}}{\partial \eta}\ ; \quad h_1 h_3 v^{(i)} = \frac{\partial \psi^{(i)}}{\partial \xi} \quad \text{for i=0,2} \tag{12}$$

Using (7) and (12)

$$curl\vec{q}^{(i)} = \left\{ \frac{1}{h_3} E^2 \psi^{(i)} \right\} \vec{e}_\phi \quad \text{for i=0,2} \tag{13}$$

$$curlcurlcurlcurl\vec{q}^{(i)} = -\frac{1}{h_1 h_2 h_3} \left\{ h_1 \frac{\partial}{\partial \eta} \left(E^4 \psi^{(i)} \right) \vec{e}_\xi - h_2 \frac{\partial}{\partial \xi} \left(E^4 \psi^{(i)} \right) \vec{e}_\eta \right\} \tag{14}$$

in which the Stokes stream function operator E^2 is given by

$$E^2 = \frac{h_3}{h_1 h_2} \left\{ \frac{\partial}{\partial \xi} \left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \eta} \right) \right\} \tag{15}$$

Using the expressions for curl $curl\ \vec{q}^{(i)}$, $curl\ curl\ curl\ curl\ \vec{q}^{(i)}$, the basic equations describing the flow in regions F_0 and F_2 are given by

$$\frac{\partial p^{(i)}}{\partial \xi} = \frac{h_1}{h_2 h_3} \left\{ -\mu \frac{\partial}{\partial \eta} \left(E^2 \psi^{(i)} \right) + \eta_1 \frac{\partial}{\partial \eta} \left(E^4 \psi^{(i)} \right) \right\}, \ i=0,2 \tag{16}$$

$$\frac{\partial p^{(i)}}{\partial \eta} = \frac{h_2}{h_1 h_3} \left\{ \mu \frac{\partial}{\partial \xi} \left(E^2 \psi^{(i)} \right) - \eta_1 \frac{\partial}{\partial \xi} \left(E^4 \psi^{(i)} \right) \right\}, i=0,2 \tag{17}$$

Eliminating $p^{(i)}$ from (16) and (17), we have

$$\left(E^6 - \frac{\lambda^2}{c^2} E^4 \right) \psi^{(i)} = 0 \quad , i=0,2 \tag{18}$$

where

$$\frac{\lambda^2}{c^2} = \frac{\mu}{\eta_1} \tag{19}$$

Thus the flow variables in the region F_0 and F_2 are completely determinable from the system of partial differential equations (18). The fluid pressure $p^{(i)}$, i=0,2 can be obtained using equations (16) and (17).

The flow in the porous region F_1 is governed by

$$div(\vec{q}^{(1)}) = 0 \tag{20}$$

$$\bar{q}^{(1)} = -\frac{k^{(1)}}{\mu} gradp^{(1)} \tag{21}$$

which implies that the pressure $p^{(1)}$ is a harmonic function given by the equation

$$\nabla^2 p^{(1)} = 0 \tag{22}$$

The determination of the relevant flow field variables $\psi^{(i)}$ and $p^{(i)}$ is subjected to the following boundary and regularity conditions.

(i) Continuity of the normal velocity components on the interfaces:
$u^{(2)} = u^{(1)}$ on S_1
$u^{(1)} = u^{(0)}$ on S_0 (23)

(ii) Tangential velocity components vanish on the interfaces:
$v^{(2)} = 0$ on S_1
$v^{(0)} = 0$ on S_0 (24)

(iii) Continuity of pressure on the interfaces:
$p^{(2)} = p^{(1)}$ on S_1
$p^{(1)} = p^{(0)}$ on S_0 (25)

(iv) $\frac{1}{2} curl\bar{q}^{(2)} = 0$ on S_1
$\frac{1}{2} curl\bar{q}^{(0)} = 0$ on S_0 (26)

(v) The velocities are regular on the axis and far away from S_1, the flow is a uniform stream which means, at infinity

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$$\psi = -\frac{1}{2}Ur^2 \tag{27}$$

Other forms of boundary conditions can also be taken. However we are using the present boundary conditions only as an initial trial for a complicated set of equations with a complicated geometry.

Since, we are dealing with a prolate coordinate system, we have

$$h_1 = h_2 = c\sqrt{(s^2 - t^2)}\,,\ h_3 = c\sqrt{(s^2-1)(1-t^2)} \tag{28}$$

$$E^2 = \frac{1}{c^2(s^2-t^2)}\left((s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial}{\partial t^2}\right) \tag{29}$$

$$\nabla^2 = \frac{1}{c^2(s^2-t^2)}\left((s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial}{\partial t^2} + 2s\frac{\partial}{\partial s} - 2t\frac{\partial}{\partial t}\right) \tag{30}$$

where

$$s = \cosh\xi;\ t = \cos\eta \tag{31}$$

III. SOLUTION FOR THE FLOW IN THE REGION Ͱ$_2$

The equation governing $\psi^{(2)}$ is

$$E^4\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi^{(2)} = 0 \tag{32}$$

The solution of equation (32) can be obtained by superposing the solutions of the equations

$$E^4\psi = 0 \tag{33}$$

and

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi = 0 \tag{34}$$

in view of the linear commutative operators E^4 and $\left(E^2 - \frac{\lambda^2}{c^2}\right)$.

Solution of equation (33):

The solution of equation (33) can be written in the form

$$\psi = \psi_0 + \psi_1 \tag{35}$$

where

$$\psi_0 = -\frac{1}{2}Uc^2(s^2-1)(1-t^2) \tag{36}$$

and

$$\psi_1 = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} G_{n+1}^{(2)}(s)P'_{n+1}(t) \tag{37}$$

where $P'_{n+1}(t)$ is the derivative of $P_{n+1}(t)$ with respect to t. The function ψ_0 in equation (36) represents the stream function due to a uniform stream of magnitude U parallel to the axis of symmetry far away from the spheroidal shell. We notice that $E^2\psi_0 = 0$ and hence $E^4\psi_0 = 0$. In view of this, ψ_1 must satisfy

$$E^4\psi_1 = 0 \tag{38}$$

It can be verified that the expression

$$f = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}^{(2)}Q'_{n+1}(s)P'_{n+1}(t) \tag{39}$$

where $Q'_{n+1}(s)$ is the derivative of Legendre function of second kind $Q_{n+1}(s)$ with respect to s, satisfies $E^2 f = 0$. In view of this, we shall impose the restriction on the functions $G_{n+1}^{(2)}(s)$ through

$$E^2\psi_1 = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}^{(2)}Q'_{n+1}(s)P'_{n+1}(t) \tag{40}$$

so that $E^4\psi_1 = 0$.

Now operating E^2 on the equation (37) and equating the result with the right hand side of equation (40), we get

$$\sum_{n=0}^{\infty}\left[\left\{(s^2-1)G_{n+1}^{(2)}(s)\right\}'' - (n+1)(n+2)\,G_{n+1}^{(2)}(s)\right]P'_{n+1}(t) = \sum_{n=0}^{\infty} A_{n+1}^{(2)}c^2(s^2-t^2)Q'_{n+1}(s)P'_{n+1}(t) \tag{41}$$

Following [2], we note that $G_{n+1}^{(2)}(s)$ is governed by the differential equation

$$(s^2-1)\left(G_{n+1}^{(2)}(s)\right)'' + 4s\left(G_{n+1}^{(2)}(s)\right)' - n(n+3)\,G_{n+1}^{(2)}(s) = g_{n+1}^{(2)}(s) \tag{42}$$

where

$$g_{n+1}^{(2)}(s) = c^2\left[\frac{(n+1)(n+2)}{(2n+3)(2n+5)}A_{n+1}^{(2)} - \frac{(n+3)(n+4)}{(2n+5)(2n+7)}A_{n+3}^{(2)}\right]Q'_{n+3}(s) \tag{43}$$
$$-c^2\left[\frac{(n-1)(n)}{(2n-1)(2n+1)}A_{n-1}^{(2)} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}A_{n+1}^{(2)}\right]Q'_{n-1}(s)$$

The equations (42) and (43) are valid for n=0,1,2,3… and the term involving $A_{-1}^{(2)}$ is to be deleted in the right hand side of equation (45) and $Q'_{-1}(s)$ is to be interpreted as $-\frac{s}{s^2-1}$ to obtain $g_1^{(2)}(s)$.

Using the method of variation of parameters, we note that

$$G_{n+1}^{(2)}(s) = \alpha_{n+1}^{(2)}P'_{n+1}(s) + B_{n+1}^{(2)}Q'_{n+1}(s) - \frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)Q'_{n+1}(s)g_{n+1}^{(2)}(s)ds$$
$$+\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)P'_{n+1}(s)g_{n+1}^{(2)}(s)ds \quad \text{for n=0,1,2…} \tag{44}$$

where s=s$_1$ represents the value specifying the outermost spheroid past which the flow is being studied. Thus the flow region Ͱ$_2$ is given by s>s$_1$. As s $\longrightarrow \infty$, $\psi^{(2)}$ must tend to 0. In view of this, we have to take $\alpha_{n+1}^{(2)} = 0$. Hence the appropriate expression for $G_{n+1}^{(2)}(s)$ is given by

$$G_{n+1}^{(2)}(s) = B_{n+1}^{(2)}Q'_{n+1}(s) - \frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)Q'_{n+1}(s)g_{n+1}^{(2)}(s)ds$$
$$+\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)P'_{n+1}(s)g_{n+1}^{(2)}(s)ds \quad \text{for n=0,1,2…} \tag{45}$$

As $g_{n+1}^{(2)}(s)$ involves one set $\left\{A_{n+1}^{(2)}\right\}$ of arbitrary constants, the functions $G_{n+1}^{(2)}(s)$ involve two sets of arbitrary constants $\left\{A_{n+1}^{(2)}\right\}$ and $\left\{B_{n+1}^{(2)}\right\}$. Using this in equation (45), we get ψ_1.

Solution of equation (34):

To solve the equation (34) (viz.) $\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi = 0$, we use the method of separation of variables, and take the solution in the form

$$\psi = c\sqrt{(s^2-1)(1-t^2)}\,R(s)S(t) \tag{46}$$

Substituting this in the equation (34), we notice that R(s) and S(t) respectively satisfy the differential equations

$$(s^2-1)R''(s) + 2s\,R'(s) - \left(\Lambda + \lambda^2 s^2 + \frac{1}{s^2-1}\right)R(s) = 0 \tag{47}$$

and

$$(1-t^2)S''(t) - 2t\,S'(t) + \left(\Lambda + \lambda^2 t^2 - \frac{1}{1-t^2}\right)S(t) = 0 \tag{48}$$

where Λ is a separation constant. [6]. These are spheroidal wave differential equations of radial and angular type respectively. To ensure regularity of solution at infinity and in the flow region we have to choose the solutions of equations (47) and (48) in the form

$$R_{1n}^{(3)}(i\lambda, s) = \left[i^{n+2}\sum_{r=0,1}^{\infty}(r+1)(r+2)d_r^{1n}(i\lambda)\right]^{-1}\left(\frac{s^2-1}{s^3}\right)^{1/2}\left(\frac{2}{\pi\lambda}\right)^{1/2}\sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d_r^{1n}(i\lambda)K_{r+3/2}(\lambda s) \tag{49}$$

and

$$S_{1n}^{(1)}(i\lambda, t) = \sum_{r=0,1}^{\infty}{}'\,d_r^{1n}(i\lambda)P_{r+1}^{(1)}(t) \tag{50}$$

where

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$$P_{r+1}^{(1)}(t) = \sqrt{1-t^2}\,\frac{d}{dt}P_{r+1}(t) \tag{51}$$

is the associated Legendre function of the first kind.

The coefficients $d_r^{1n}(i\lambda)$ in the above expansions are constants depending on the parameter $i\lambda$ and the suffix r has the value 1,3,5… or 0, 2,4,6… depending upon the odd or even values of n+1. We have therefore the solution

$$\psi_2 = c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n^{(2)} R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t) \tag{52}$$

where $C_n^{(2)}$'s are constants.

Hence, the stream function for the region F_2 is given by

$$\psi^{(2)}(s,t) = -\frac{1}{2}Uc^2(s^2-1)(1-t^2) + c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} G_{n+1}^{(2)}(s)P'_{n+1}(t) + c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n^{(2)} R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t) \tag{53}$$

We can see that

$$E^2\psi^{(2)} = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}^{(2)} Q'_{n+1}(s)P'_{n+1}(t) + \frac{\lambda^2}{c}\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n^{(2)} R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t) \tag{54}$$

and

$$E^4\psi^{(2)} = \frac{\lambda^4}{c^3}\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n^{(2)} R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t) \tag{55}$$

which are recorded for future use.

IV. Pressure distribution in F_2

The equations (16) and (17) for i = 2 and using equation (37) are

$$\frac{\partial p^{(2)}}{\partial s} = \frac{\mu}{2c(s^2-1)}\frac{\partial}{\partial t}\left(E^2\psi^{(2)}\right) - \frac{\eta_1}{2c(s^2-1)}\frac{\partial}{\partial t}\left(E^4\psi^{(2)}\right) \tag{56}$$

and

$$\frac{\partial p^{(2)}}{\partial t} = -\frac{\mu}{2c(1-t^2)}\frac{\partial}{\partial s}\left(E^2\psi^{(2)}\right) + \frac{\eta_1}{2c(1-t^2)}\frac{\partial}{\partial s}\left(E^4\psi^{(2)}\right) \tag{57}$$

Using the expressions in equations (54) and (55) in (56) and (57), on integration we get

$$p^{(2)}(s,t) = -\mu c\sum_{n=0}^{\infty} A_{n+1}^{(2)}(n+1)(n+2)Q_{n+1}(s)P_{n+1}(t) \tag{58}$$

Thus $\psi^{(2)}(s,t)$ and $p^{(2)}(s,t)$ given in equations (53) and (58) respectively are the stream function and pressure distribution for the region F_2. These involve the three sets of constants $\{A_n^{(2)}\}, \{B_n^{(2)}\}, \{C_n^{(2)}\}$ as can be seen from equations (53),(44) and (58).

V. SOLUTION FOR THE FLOW IN THE REGION F_0

The equation for $\psi^{(0)}$ is given by equations

$$E^4(E^2 - \frac{\lambda^2}{c^2})\psi^{(0)} = 0 \tag{59}$$

Solution of equation (59) can be obtained by superimposing the solutions of $E^4\psi = 0$ and $\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi = 0$. The procedure to obtain the solution can be carried out exactly on similar lines as in the case of F_2 with the difference that here s is bounded ($1<s<s_0$) and origin and a part of the axis of symmetry are in the flow field. Carrying out the procedure exactly as in the case of F_2, we get the stream function suitable for F_0 as

$$\psi^{(0)} = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} G_{n+1}^{(0)}(s)P'_{n+1}(t) + c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} D_n^{(0)} R_{1n}^{(4)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t) \tag{60}$$

where

$$G_{n+1}^{(0)}(s) = B_{n+1}^{(0)}P'_{n+1}(s) - \frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_s^{s_0}(s^2-1)Q'_{n+1}(s)g_{n+1}^{(0)}(s)ds \quad \frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_s^{s_0}(s^2-1)P'_{n+1}(s)g_{n+1}^{(0)}(s)ds \tag{61}$$

with

$$g_{n+1}^{(0)}(s) = c^2\left[\frac{(n+1)(n+2)}{(2n+3)(2n+5)}A_{n+1}^{(0)} - \frac{(n+3)(n+4)}{(2n+5)(2n+7)}A_{n+3}^{(0)}\right]P'_{n+3}(s) - c^2\left[\frac{(n-1)(n)}{(2n-1)(2n+1)}A_{n-1}^{(0)} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}A_{n+1}^{(0)}\right]P'_{n-1}(s) \tag{62}$$

Here again (62) is valid for n=0,1,2… with the understanding that the term $A_{-1}^{(0)}$ is to be deleted when we take n=0 and $P'_{-1}(t)$ is to be interpreted as 0.

A comment at this stage is in order. Comparing the expression for $g_{n+1}^{(0)}(s)$ that we get here and the expression for $g_{n+1}^{(2)}(s)$ of equation (43) we obtained in the case of F_2, we notice that here we have $P'_{n+3}(s)$ and $P'_{n-1}(s)$ respectively in place of $Q'_{n+3}(s)$ and $Q'_{n-1}(s)$ there in equation (43). This selection of the appropriate Legendre functions is guided by the bounded nature of the flow regime F_0 and the unbounded nature of the flow regime F_2.

Further, the function $R_{1n}^{(4)}(i\lambda, s)$ given in equation (60) is a radial spheroidal wave function given by

$$R_{1n}^{(4)}(i\lambda, s) = \left[i^{n-1}\sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d_r^{1n}(i\lambda)\right]^{-1}\left(\frac{s^2-1}{s^3}\right)^{1/2} \left(\frac{2}{\pi\lambda}\right)^{1/2}\sum_{r=0,1}^{\infty}{}'(-1)^{r+1}(r+1)(r+2)d_r^{1n}(i\lambda)K_{r+3/2}(-\lambda s) \tag{63}$$

Evaluating $E^2\psi^{(0)}$, $E^4\psi^{(0)}$ and using their expressions in equation (60), we note that pressure distribution in F_0 is given by

$$p^{(0)}(s,t) = -\mu c\sum_{n=0}^{\infty} A_{n+1}^{(0)}(n+1)(n+2)P_{n+1}(s)P_{n+1}(t) \tag{64}$$

Thus and stream function $\psi^{(0)}(s,t)$ and pressure distribution $p^{(0)}(s,t)$ for the flow regime F_0 are given by equations (60) and (64) respectively. These involve three sets of constants $\{A_n^{(0)}\}$, $\{B_n^{(0)}\}$, $\{D_n^{(0)}\}$ as can be seen from equations (60),(61) and (64).

VI. SOLUTION FOR THE FLOW IN THE REGION F_1

We have seen earlier that the flow in the porous region $F1$ is governed by the equations (20) and (21) which lead to the equation (22). The equation (22) implies that the pressure distribution $p^{(1)}(s,t)$ in F_1 is harmonic and hence it is given by

$$p^{(1)}(s,t) = \sum_{n=1}^{\infty}(\alpha_n P_n(s) + \beta_n Q_n(s))P_n(t) \tag{65}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ constitute another set of arbitrary constants to be determined. The velocity components $u^{(1)}(s,t)$ and $v^{(1)}(s,t)$ can be determined from equations (21) and (65).

Thus, in all, we have eight sets of unknown constants $\{A_n^{(2)}\}, \{B_n^{(2)}\}, \{C_n^{(2)}\}, \{A_n^{(0)}\}, \{B_n^{(0)}\}, \{D_n^{(0)}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and

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these can be determined by using the eight boundary conditions given by the equations (23),(24),(25) and (26).

VII. VELOCITY COMPONENTS IN THE REGIONS Ғ$_0$, Ғ$_1$, Ғ$_2$

The velocity components can be obtained by using the expressions for $\psi^{(0)}$ and $\psi^{(2)}$ given in equations (53), (60) and $p^{(1)}$ given in equation (65). Thus the expressions for the velocity components $u^{(2)}$, $v^{(2)}$; $u^{(0)}$, $v^{(0)}$; $u^{(1)}$, $v^{(1)}$ can be written explicitly. Using these expressions and those of $p^{(0)}$ and $p^{(2)}$ in the boundary conditions given by equations (23), (24), (25) and (26), we can write the equations that lead to the determination ofthe arbitrary constants.

VIII. DETERMINATION OF ARBITRARY CONSTANTS

In view of the equations (23)-(27) and the orthogonality property of Legendre functions and the associated Legendre functions, we have

$$Uc^2\left(s_1^2-1\right)\delta_{n0}-c^2\left({s_1}^2-1\right)B_{n+1}^{(2)}Q'_{n+1}(s_1)(n+1)(n+2)-c(n+1)(n+2)\sum_{m=1}^{\infty}C_m^{(2)}\sqrt{{s_1}^2-1}R_{1n}^{(3)}(i\lambda,s_1)d_n^{1m}(i\lambda)=-k^{(1)}c\left(s_1^2-1\right)\left(\alpha_{n+1}P'_{n+1}(s_1)+\beta_{n+1}Q'_{n+1}(s_1)\right) \tag{66}$$

$$c^2\left({s_0}^2-1\right)B_{n+1}^{(0)}P'_{n+1}(s_0)(n+1)(n+2)+c\sum_{m=1}^{\infty}D_m^{(0)}\sqrt{{s_0}^2-1}R_{1n}^{(4)}(i\lambda,s_0)d_n^{1m}(i\lambda)(n+1)(n+2)=k^{(1)}c\left(s_0^2-1\right)\left(\alpha_{n+1}P'_{n+1}(s_0)+\beta_{n+1}Q'_{n+1}(s_0)\right) \tag{67}$$

$$-Uc^2s_1\delta_{0n}+c^2B_{n+1}^{(2)}(n+1)(n+2)Q_{n+1}(s_1)+c\sum_{m=1}^{\infty}C_m^{(2)}\frac{d}{ds}\left[\sqrt{s^2-1}R_{1n}^{(3)}(i\lambda,s)\right]_{ons=s_1}d_n^{1m}(i\lambda)=0 \tag{68}$$

$$c^2B_{n+1}^{(0)}(n+1)(n+2)P_{n+1}(s_0)+c\sum_{m=1}^{\infty}D_m^{(0)}\frac{d}{ds}\left[\sqrt{s^2-1}R_{1n}^{(4)}(i\lambda,s)\right]_{ons=s_0}d_n^{1m}(i\lambda)=0 \tag{69}$$

$$c\sqrt{({s_1}^2-1)}A_{n+1}^{(2)}Q'_{n+1}(s_1)+\frac{\mu}{\eta_1}\sum_{m=1}^{\infty}C_m^{(2)}R_{1m}^{(3)}(i\lambda,s_1)d_n^{1m}(i\lambda)=0 \tag{70}$$

$$c\sqrt{({s_0}^2-1)}A_{n+1}^{(0)}P'_{n+1}(s_0)+\frac{\mu}{\eta_1}\sum_{m=1}^{\infty}D_m^{(0)}R_{1m}^{(4)}(i\lambda,s_0)d_n^{1m}(i\lambda)=0 \tag{71}$$

$$-\mu cA_{n+1}^{(2)}Q_{n+1}(s_1)(n+1)(n+2)=\left(\alpha_{n+1}P_{n+1}(s_1)+\beta_{n+1}Q_{n+1}(s_1)\right) \tag{72}$$

$$-\mu cA_{n+1}^{(0)}P_{n+1}(s_0)(n+1)(n+2)=\left(\alpha_{n+1}P_{n+1}(s_0)+\beta_{n+1}Q_{n+1}(s_0)\right) \tag{73}$$

Determination of the arbitrary constants in the problem, as it stands, is quite complicated, but of course, is not unsurmountable. Eliminating the constants we see that $\left\{C_m^{(2)}\right\}$and $\left\{D_m^{(0)}\right\}$ have to simultaneously satisfy a system of infinite non homogeneous system of linear equations.

It is heartening to note that as the coefficients $d_n^{1m}(i\lambda)$ are zero for odd values of m+1+n, we can segregate the above system of equations in $\left\{C_m^{(2)}\right\}$and $\left\{D_m^{(0)}\right\}$ into two sub systems containing $\left\{C_{2m+1}^{(2)}\right\}$, $\left\{D_{2m+1}^{(0)}\right\}$ and $\left\{C_{2m}^{(2)}\right\}$ $\left\{D_{2m}^{(0)}\right\}$. The sub system involving $\left\{C_{2m}^{(2)}\right\}$and $\left\{D_{2m}^{(0)}\right\}$ is seen to be homogeneous and in view of this $C_{2m}^{(2)}=D_{2m}^{(0)}=0$ for all positive integral values of m. Finally we end up with the following non homogeneous system. After a straight forward but lengthy algebra

$$\sum_{m=1}^{\infty}{}'\left(A_{nm}C_m^{(2)}+B_{nm}D_m^{(0)}\right)=-Uc\delta_{0n} \tag{74}$$

and

$$\sum_{m=1}^{\infty}{}'\left(A_{nm}^*C_m^{(2)}+B_{nm}^*D_m^{(0)}\right)=0 \tag{75}$$

where

$$A_{nm}=d_{2n}^{1m}(i\lambda)\left[\begin{array}{l}\left\{\sqrt{{s_1}^2-1}\frac{d}{ds}R_{1m}^{(3)}(i\lambda,s_1)+\frac{s_1}{\sqrt{{s_1}^2-1}}R_{1m}^{(3)}(i\lambda,s_1)\right\}({s_1}^2-1)Q'_{2n+1}(s_1)\\ -2(n+1)(2n+1)\sqrt{{s_1}^2-1}Q_{2n+1}(s_1)R_{1m}^{(3)}(i\lambda,s_1)\\ +\frac{(n+1)(2n+1)}{P_{2n+1}(s_1)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q_{2n+1}(s_1)}2\frac{\mu}{\eta_1}k^{(1)}({s_1}^2-1)\\ \frac{Q_{2n+1}(s_1)}{Q'_{2n+1}(s_1)}\frac{Q_{2n+1}(s_1)}{\sqrt{{s_1}^2-1}}R_{1m}^{(3)}(i\lambda,s_1)*\left\{P'_{2n+1}(s_1)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q'_{2n+1}(s_1)\right\}\end{array}\right] \tag{76}$$

$$B_{nm}=-d_{2n}^{1m}(i\lambda)\left[\begin{array}{l}\frac{(n+1)(2n+1)}{P_{2n+1}(s_1)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q_{2n+1}(s_1)}\frac{\mu}{\eta_1}2k^{(1)}({s_1}^2-1)\\ \frac{Q_{2n+1}(s_1)}{P'_{2n+1}(s_0)}\frac{P_{2n+1}(s_0)}{\sqrt{{s_0}^2-1}}R_{1m}^{(4)}(i\lambda,s_0)*\left\{P'_{2n+1}(s_1)Q_{2n+1}(s_1)-P_{2n+1}(s_1)Q'_{2n+1}(s_1)\right\}\end{array}\right] \tag{77}$$

$$A_{nm}^*=d_{2n}^{1m}(i\lambda)\left[\begin{array}{l}\frac{(n+1)(2n+1)}{P_{2n+1}(s_1)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q_{2n+1}(s_1)}\frac{\mu}{\eta_1}2k^{(1)}({s_0}^2-1)\\ \frac{Q_{2n+1}(s_1)}{Q'_{2n+1}(s_1)}\frac{P_{2n+1}(s_0)}{\sqrt{{s_1}^2-1}}R_{1m}^{(3)}(i\lambda,s_1)*\left\{P'_{2n+1}(s_0)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q'_{2n+1}(s_0)\right\}\end{array}\right] \tag{78}$$

$$B_{nm}^*=d_{2n}^{1m}(i\lambda)\left[\begin{array}{l}2(n+1)(2n+1)\sqrt{{s_0}^2-1}P_{2n+1}(s_0)R_{1m}^{(4)}(i\lambda,s_0)\\ -\left\{\sqrt{{s_0}^2-1}\frac{d}{ds}R_{1m}^{(4)}(i\lambda,s_0)+\frac{s_0}{\sqrt{{s_0}^2-1}}R_{1m}^{(4)}(i\lambda,s_0)\right\}({s_0}^2-1)P'_{2n+1}(s_0)\\ +\frac{(n+1)(2n+1)}{P_{2n+1}(s_1)Q_{2n+1}(s_0)-P_{2n+1}(s_0)Q_{2n+1}(s_1)}\frac{\mu}{\eta_1}2k^{(1)}({s_0}^2-1)\\ \frac{P_{2n+1}(s_0)}{P'_{2n+1}(s_0)}\frac{P_{2n+1}(s_0)}{\sqrt{{s_0}^2-1}}R_{1m}^{(4)}(i\lambda,s_0)*\left\{Q'_{2n+1}(s_0)P_{2n+1}(s_1)-Q_{2n+1}(s_1)P'_{2n+1}(s_0)\right\}\end{array}\right] \tag{79}$$

As an analytical determination of $C_{2m+1}^{(2)}, D_{2m+1}^{(0)}$ is not possible, we have to necessarily resort to a numerical determination of the constants. For this, we truncate the two systems (74) and (75) so as to give a 10 by 10 system with $C_1^{(2)}$, $C_3^{(2)}$ … $C_9^{(2)}$ and $D_1^{(0)}, D_3^{(0)} \ldots D_9^{(0)}$ After determining these, it is possible to evaluate numerically the other constants. The details of the manipulations are omitted in view of the lengthiness of the expressions and the final system only is reported here.

IX. DETERMINATION OF DRAG

The drag D can be written in the form

$$D=2\pi c^2\sqrt{{s_1}^2-1}\int_{-1}^{1}\left(t\sqrt{s^2-1}\,\mathrm{t}_{\xi\xi}-s\sqrt{1-t^2}\,\mathrm{t}_{\xi\eta}\right)_{s=s_1}dt \tag{80}$$

and this simplifies to

$$2\pi c^2\sqrt{{s_1}^2-1}\left(\begin{array}{l}\int_{-1}^{1}t\sqrt{{s_1}^2-1}\,p^{(2)}(s_1,t)dt+\\ \frac{4s_1({s_1}^2-1)^{3/2}k^{(1)}}{c^2}\sum_{n=0}^{\infty}\left(\alpha_{n+1}P'_{n+1}(s)+\beta_{n+1}Q'_{n+1}(s_1)\right)\int_{-1}^{1}\frac{tP_{n+1}(t)}{({s_1}^2-t^2)^2}dt-\\ \frac{2\mu}{c^2}s_1\sqrt{{s_1}^2-1}\,\mathrm{k}^{(1)}\sum_{n=0}^{\infty}\left(\alpha_{n+1}P'_{n+1}(s)+\beta_{n+1}Q'_{n+1}(s_1)\right)\int_{-1}^{1}\frac{(1-t^2)P'_{n+1}(t)}{({s_1}^2-t^2)}dt-\\ \mu s_1\frac{\lambda^2}{c^2}\sum_{n=0}^{\infty}C_n^{(2)}R_{1n}^{(3)}(i\lambda,s_1)\int_{-1}^{1}\sqrt{(1-t^2)}S_{1n}^{(1)}(i\lambda,t)dt\end{array}\right) \tag{81}$$

Using the relations from Hobson[7], given by

$$\int_{-1}^{1}\frac{(1-t^2)P'_n(t)}{({s_1}^2-t^2)}dt=-\frac{2}{s_1}({s_1}^2-1)Q'_n(s_1) \tag{82}$$

and

$$\int_{-1}^{1}\frac{tP_n(t)}{({s_1}^2-t^2)^2}dt=-\frac{1}{s_1}Q'_n(s_1), \tag{83}$$

the drag simplifies to

$$D = 2\pi c^2 \sqrt{s_1^2 - 1}\left(\begin{array}{l} \dfrac{4\mu c}{3} A_1^{(2)} \sqrt{s_1^2 - 1}\left(Q_1(s_1) - s_1 Q_1'(s_1)\right) \\ -\dfrac{4}{3}\mu \dfrac{\lambda^2}{c^2} s_1 \sum_{n=0}^{\infty}{}' C_n^{(2)} R_{1n}^{(3)}(i\lambda, s_1) d_0^{1n}(i\lambda) \end{array} \right) \quad (84)$$

After further simplification we see that the drag due to the surface stress is given by D = $1/3A_1^{(2)}$ (85)

where $A_1^{(2)} = -\dfrac{2(\mu + k)}{k}\lambda^2 \dfrac{\sum_{m=1}^{\infty}{}' C_m^{(2)} R_{1m}^{(3)}(i\lambda, s_1) d_0^{1m}(i\lambda)}{\sqrt{s_1^2 - 1}\,Q_1'(s_1)}$ (86)

We refer to $1/3A_1^{(2)}$ as the non dimensional drag and this depends upon the eccentricity of the spheroid, the micro polarity parameter, an additional material constant λ and the permeability parameter $k^{(1)}$.

X. NUMERICAL DISCUSSION

The drag on the spheroidal shell is numerically evaluated for several parameter values and the variation of the drag is displayed through figures (1) to (3).

For each value of the permeability parameter kp, the drag is increasing as λ increases. An increase in λ implies a decrease in the couple stress viscosity η. Hence, we notice that as resistance to rotation decreases, the body experiences a greater drag. For a fixed λ, for an increase in kp, the drag is seen to be slightly increasing (see fig(1)) and the increase is not significant.

An increase in s_1 indicates an increase in the size of the outer spheroid. The fig (2) shows that as the size of the outer spheroid increases, for a fixed λ, when the size of the inner spheroid is fixed, the drag is increasing. Further as the parameter λ increases for a fixed s_1, the drag increases.

In fig (3), we plotted the variation of drag for fixed values of kp and s_0, with respect to varying s_1 and diverse values of λ. Here also we note that as the size of the outer spheroid increases, the drag increases. Also as the couple stress parameter λ increases, the drag is significantly influenced.

We tried to plot the streamline pattern for different values of λ, kp with s_0= 1.5 and s_1 = 2.0. Here also the streamline pattern is similar to the one obtained by the authors in the case of flow of a micropolar fluid past a porous spheroidal shell [8]. The streamline pattern for the outer region as well as the porous region is slightly disturbed where as for the fluid core region there is a considerable disturbance and the patterns are similar to the ones obtained by Happel and Brenner (see page 129 of [9]).

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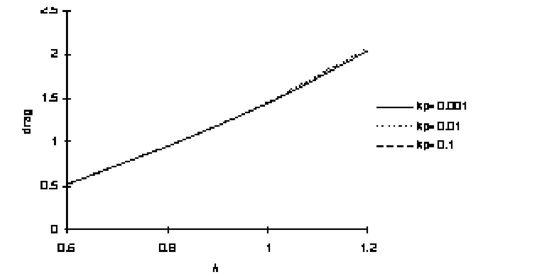
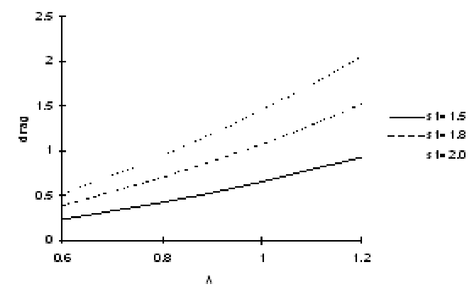
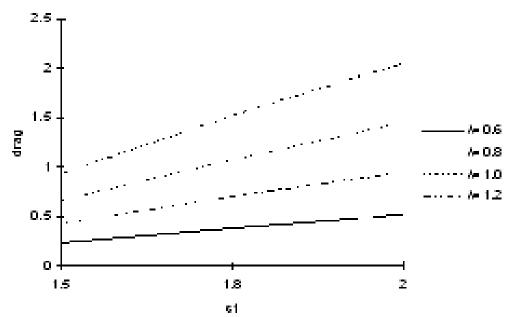


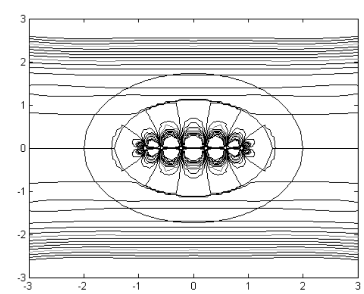
Fig(1): Variation of drag with respect to λ for different kp when s_0=1.2 and s_1=2.0



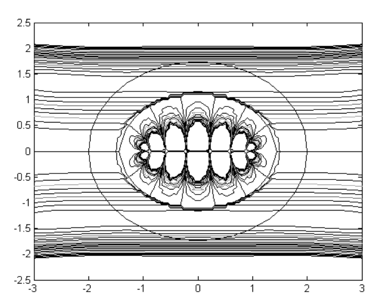
Fig(2): Variation of drag with respect to λ for different s_1 when s_0=1.2 , kp=0.01



Fig(3):Variation of drag with respect to s1for different λ when s_0=1.2 ,kp=0.01



Fig(4): Streamline pattern for λ =1.2 and kp-0.0005



Fig(5): Streamline pattern for λ =1.5 and kp-0.0005

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