Fuzzy Random Linear Optimization under Possibilistic Downside Risk Measures: Minimization of Possibilistic Low Partial Moment

Hideki Katagiri

Abstract—**This paper considers new downside risk-aversion models for linear optimization (linear programming) with discrete fuzzy random variables. Through new downside risk measures for fuzzy stochastic optimization problems, possibilistic low partial moment (PLPM) models are constructed by incorporating possibility and necessity measures into classical low partial moment. To provide practical models, the case of linear membership functions is focused on. It is shown that the problems involving both fuzziness and randomness are transformed into deterministic polynomial optimization problems.**

Index Terms—**Linear programming, fuzzy random variable, downside risk measure, possibilistic lower partial moment, polynomial optimization problem**

I. Introduction

Simultaneous consideration of fuzziness and randomness is highly important in modeling decision making problems, because decision making by humans in stochastic environments is intrinsically based not only on randomness but also on fuzziness. One of traditional tools for taking into consideration uncertainty of parameters involved in mathematical optimization problems is stochastic programming [2]. On the other hand, to deal with human judgments and/or knowledge in decision making, fuzzy optimization frameworks involving fuzzy programming [38] and possibilistic programming [8] have been developed based on fuzzy-set theory [36] and possibility theory [3], respectively.

In the last decade, mathematical optimization models in decision making which take into consideration both fuzziness and randomness have considerably drawn attentions in the research fields such as linear programming [1], [13], [18], [20], [35], integer programming [17], transportation [6], facility layout [32], network optimization [14], [10] and portfolio selection [16].

Previous studies on fuzzy random linear programming problems have mainly focused on the case where the coefficients of the objective function and the constraints are expressed by *continuous* fuzzy random variables. Linear programming problems with *discrete* fuzzy random variables [12], [17] have not fully been discussed so far. This is the motivation of this article.

In this paper, we provide new downside risk measures for fuzzy random optimization, which are constructed by incorporating possibility theory into classical low partial moment (LPM). We formulate new downside risk-aversion models, called *possibilistic low partial moment* models, for linear programming (linear optimization) problems with discrete fuzzy random variable. It is shown that the formulated problems based on the newly proposed risk measures are transformed into polynomial optimization problem [26].

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II. Preliminaries

Fuzzy random programming, which means mathematical programming in which *fuzzy random variables* are involved, has been developed as one of the most well-known and popular techniques for fuzzy stochastic optimization in decision making. From mathematical viewpoints, there are mainly two definitions of fuzzy random variables. A fuzzy random variable was firstly defined by Kwakernaak [24] in 1978 as random variables whose realized values for given events are not real but fuzzy numbers. Kruse [23] provided some expanded concepts of the model similar to the fuzzy random variable defined by Kwakernaak. Puri and Ralescu [33] defined fuzzy random variables as random fuzzy sets and developed a mathematical basis of fuzzy random variables [22]. Overviews of fuzzy random variables were also presented in some articles [5], [34].

We introduce a general definition of fuzzy random variables, which is based on previous works [4], [23], [24]:

Definition 1: (*Fuzzy random variable*)
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $F(\mathbb{R})$ denote the set of all fuzzy numbers in $\mathbb{R}$, where $F(\mathbb{R})$ denotes a class of normal convex fuzzy subsets of $\mathbb{R}$ having compact α level set for $\alpha \in [0,1]$. A fuzzy random variable is a mapping $\tilde{\bar{A}}:\ \Omega \to F(\mathbb{R})$ such that for any $\alpha \in [0,1]$ and all $\omega \in \Omega$, the real-valued mapping

$$\inf \tilde{\bar{A}}_\alpha :\ \Omega \to \mathbb{R},\ \text{satisfying}\ \inf \tilde{A}_\alpha(\omega) = \inf(\tilde{A}(\omega))_\alpha$$

and

$$\sup \tilde{\bar{A}}_\alpha :\ \Omega \to \mathbb{R},\ \text{satisfying}\ \sup \tilde{A}_\alpha(\omega) = \sup(\tilde{A}(\omega))_\alpha$$

are real-valued random variables, that is, Borel measurable real-valued functions. $(\tilde{A}(\omega))_\alpha$ is a nonempty compact interval defined by

$$(\tilde{A}(\omega))_\alpha = \begin{cases} \{x \in \mathbb{R} \mid \mu_{\tilde{A}(\omega)}(x) \geq \alpha\} & if\ \alpha \in (0,1] \\ \mathrm{cl}(supp\ \mu_{\tilde{A}(\omega)}) & if\ \alpha = 0, \end{cases}$$

where $\mu_{\tilde{A}(\omega)}$ is the membership function of a fuzzy set $\tilde{A}(\omega)$, and $cl(supp\ \mu_{\tilde{A}(\omega)})$ denotes the closure of set $supp\ \mu_{\tilde{A}(\omega)}$, and $supp\ \mu_{\tilde{A}(\omega)}$ denotes a support of function $\mu_{\tilde{A}(\omega)}$.

Fuzzy random optimization models were firstly developed by Luhandjula and his research colleagues [29], [31] as linear programming problems with fuzzy random variable

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coefficients, and further studied by other researchers [13], [18], [27], [28], [35]. Major fuzzy stochastic programming models were discussed in the survey paper [30].

For the purpose of applying fuzzy random variables to decision making problems, Katagiri et al. [13], [14], [15], [16], [19], [18] introduced some special types of fuzzy random variables. To systematically discuss these papers from the viewpoint of the types of fuzzy random variables, we define an *L-R fuzzy random variable* as follows:

Definition 2: (*L-R fuzzy random variable*)
Let $\bar{d}$, $\bar{\beta}$ and $\bar{\gamma}$ be random variables of which realization for a given event $\omega \in \Omega$ are $d(\omega)$, $\beta(\omega)$ and $\gamma(\omega)$, respectively, where Ω is a sample space, and $\beta(\omega)$ and $\gamma(\omega)$ are positive constants for any $\omega \in \Omega$. Then, a fuzzy random variable $\tilde{\bar{F}}$ is said to be an *L-R fuzzy random variable*, denoted by $(\bar{d}, \bar{\beta}, \bar{\gamma})_{LR}$, if its realized values $\tilde{F}(\omega) = (d(\omega), \beta(\omega), \gamma(\omega))_{LR}$ for any event $\omega \in \Omega$ are L-R fuzzy numbers defined as

$$\mu_{\tilde{F}(\omega)}(\tau) = \begin{cases} L\left(\dfrac{d(\omega)-\tau}{\beta(\omega)}\right) & \text{if } \tau \le d(\omega) \\ R\left(\dfrac{\tau-d(\omega)}{\gamma(\omega)}\right) & \text{if } \tau > d(\omega). \end{cases} \tag{1}$$

L-R fuzzy random variables were introduced to decision making problems such as a portfolio selection problem [16], a linear programming problem [15] and a multi-objective programming problem [13], [18]. As a special type of L-R fuzzy random variables, an *L fuzzy random variable* can be considered when $L = R$ holds, which means that the left-hand and the right-hand sides of reference functions are the same. L fuzzy random variables were considered in network optimization problems such as bottleneck minimum spanning tree problems [9], [14]. The L fuzzy random variable is closely related to a so-called "hybrid number" which was originally introduced by [21]. When $L(t) = R(t) = \max\{0, 1-|t|\}$ in Definition 2, we call such an L-R fuzzy random variable a *triangular fuzzy random variable*. Triangular fuzzy random variables were introduced in the previous study on a multi-objective linear programming problem [19], where spread parameters $\bar{\beta}$ and $\bar{\gamma}$ are not random variables but constant values.

III. Discrete fuzzy random variable

The concept of *discrete fuzzy random variable* was originally defined by Kwakernaak [25]. In this paper, we provide the definition of discrete fuzzy random variable as follows:

Definition 3: (*Discrete fuzzy random variable*)
Let Ω be a set of events such that the occurrence probability of each event $\omega_k \in \Omega$ is p_k and that $\sum_k p_k = 1$. Let $\tilde{F}_k$ be a fuzzy set characterized by a membership function $\mu_{\tilde{F}_k}$, and let $\mathcal{F}$ be a set of F_k, $\forall k \in K$, where K is an index set of k. Let $\tilde{\bar{F}}$ be a mapping from Ω to $\mathcal{F}$ such that $\tilde{\bar{F}}(\omega_k) \triangleq \tilde{F}_k$. Then, a mapping $\tilde{\bar{F}}$ is said to be *a discrete fuzzy random variable*.

As a special type of discrete fuzzy random variables, we call a discrete fuzzy random variable a *discrete L-R fuzzy random variable* when each realization $\tilde{F}_k$ is an L-R number in Definition 3 as follows:

$$\mu_{\tilde{F}_k}(\tau) = \begin{cases} L\left(\dfrac{d_k-\tau}{\beta_k}\right) & \text{if } \tau \le d_k \\ R\left(\dfrac{\tau-d_k}{\gamma_k}\right) & \text{if } \tau > d_k. \end{cases} \tag{2}$$

In particular, when $L(t) = R(t) = \max\{0, 1-|t|\}$ holds in Definition 2, we call such a discrete L-R fuzzy random variable a *discrete triangular fuzzy random variable*. In other words, a discrete triangular fuzzy random variable $\tilde{\bar{F}}$ is a discrete fuzzy random variable whose realization for each event ω_k is a triangular fuzzy number characterized by the following membership function:

$$\mu_{\tilde{F}_k}(\tau) = \begin{cases} \max\left\{1-\dfrac{|d_k-\tau|}{\beta_k}, 0\right\} & \text{if } \tau \le d_k \\ \max\left\{1-\dfrac{|\tau-d_k|}{\gamma_k}, 0\right\} & \text{if } \tau > d_k. \end{cases} \tag{3}$$

Discrete fuzzy random variables were firstly introduced in the previous study on a linear programming problem [12], and further discussed in a network optimization problem [11] and a multi-objective 0-1 programming problem [17]. These previous studies mainly focused on the decision making models which maximize probabilistic expectation of the possibility that the goal is attained.

To the author's best knowledge, there are few studies which considers downside risk-aversion models for linear programming with discrete fuzzy random variables, although possibilistic value-at-risk (PVaR) models [10] were constructed by the author and his colleagues. Some previous studies discussed variance models [12] and expectation models [11], [17].

Under these circumstances, we propose a new downside risk measure in fuzzy stochastic decision making environments, called *possibilistic low partial moment* (PLPM), which is an extended concept of classical low partial moment (LPM) through the incorporation of possibility theory into the LPM.

IV. Problem formulation

Assuming that the coefficients of the objective functions are given as discrete fuzzy random variables, we consider the following fuzzy random programming problem:

$$\left.\begin{array}{ll} \text{minimize} & \tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}, \ l = 1, 2, \ldots, q \\ \text{subject to} & A\boldsymbol{x} \le \boldsymbol{b}, \ \boldsymbol{x} \ge \boldsymbol{0}, \end{array}\right\} \tag{4}$$

where $\tilde{\bar{\boldsymbol{C}}}_l = (\tilde{\bar{C}}_{l1}, \ldots, \tilde{\bar{C}}_{ln}), \ l = 1, 2, \ldots, q$ is an n dimensional coefficient row vector of which elements are discrete fuzzy random variables, $\boldsymbol{x}$ is an n dimensional decision variable column vector, A is an $m \times n$ coefficient matrix, and $\boldsymbol{b}$ is an m dimensional column vector. When the number of objective functions equals to 1 ($q = 1$), then problem (4) becomes a single-objective fuzzy random programming problem; otherwise, when $q \ge 2$, (4) is a multi-objective fuzzy random programming problem.

In problem (4), each element $\tilde{\bar{C}}_{lj}$ of the coefficient vector $\tilde{\bar{\boldsymbol{C}}}_l = (\tilde{\bar{C}}_{l1}, \ldots, \tilde{\bar{C}}_{ln}), \ l = 1, 2, \ldots, q$ in (4) is a discrete triangular fuzzy random variable whose realization is a triangular fuzzy number $\tilde{C}_{ljk} = (d_{ljk}, \beta_{ljk}, \gamma_{ljk})_{tri}$ for $\omega_{lk} \in \Omega_l$,

$l = 1, 2, \ldots, q$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r_l$, where an event ω_{lk} occurs at probability p_{lk} and $\sum_k p_{lk} = 1$. The membership function of $\tilde{C}_{ljk}$ is given as follows:

$$\mu_{\tilde{C}_{ljk}}(\tau) = \begin{cases} \max\left\{1 - \dfrac{|d_{ljk} - \tau|}{\beta_{ljk}}, 0\right\} & \text{if } \tau \leq d_{ljk} \\ \max\left\{1 - \dfrac{|\tau - d_{ljk}|}{\gamma_{ljk}}, 0\right\} & \text{if } \tau > d_{ljk}. \end{cases} \tag{5}$$

Then, by applying the Zadeh's extension principle [36], the realized value of each objective function $\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}$ for a given event ω_{lk} which occurs at probability p_{lk} is calculated as a single triangular fuzzy number $(\boldsymbol{d}_{lk}\boldsymbol{x}, \boldsymbol{\beta}_{lk}\boldsymbol{x}, \boldsymbol{\gamma}_{lk}\boldsymbol{x})_{tri}$, $l = 1, 2, \ldots, q$, $k = 1, 2, \ldots, r_l$, which is characterized by

$$\mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}(\upsilon) = \begin{cases} \max\left\{1 - \dfrac{|\boldsymbol{d}_{lk}\boldsymbol{x} - \upsilon|}{\boldsymbol{\beta}_{lk}\boldsymbol{x}}, 0\right\} & \text{if } \upsilon \leq \boldsymbol{d}_{lk}\boldsymbol{x} \\ \max\left\{1 - \dfrac{|\upsilon - \boldsymbol{d}_{lk}\boldsymbol{x}|}{\boldsymbol{\gamma}_{lk}\boldsymbol{x}}, 0\right\} & \text{if } \upsilon > \boldsymbol{d}_{lk}\boldsymbol{x}. \end{cases} \tag{6}$$

Figures 1 and 2 show the membership functions of $\tilde{C}_{ljk}$ and $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$.

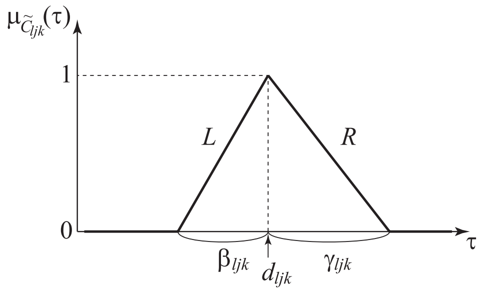

Fig. 1. Membership function $\mu_{\tilde{C}_{ljk}}$ of the realized value $\tilde{C}_{ljk}$ for the kth event of a discrete triangular fuzzy random variable $\tilde{\tilde{C}}_{lj}$

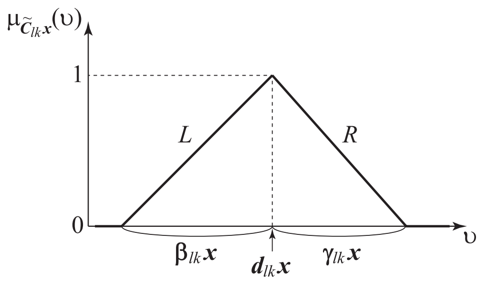

Fig. 2. Membership function $\mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}$ of the realized value $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$ for the kth event of a discrete triangular fuzzy random variable $\tilde{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}$

In decision making situations where the objective function is to be minimized, decision makers (DMs) often have fuzzy goals such as "the objective function value $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$ is substantially less than or equal to a certain value f_l," which is expressed by $\tilde{\tilde{\boldsymbol{C}}}_{lk}\boldsymbol{x} \lesssim f_l$, where $\lesssim$ denotes "substantially less than or equal to." Let $\mu_{\tilde{G}_l}$ be a membership function of fuzzy set $\tilde{G}_l$ such that the degree of y being substantially less than or equal to a certain value f_l is represented with $\mu_{\tilde{G}_l}(y)$.

Here, we focus on a case where all the membership functions of fuzzy numbers and fuzzy goals are represented by *linear membership functions* which are the following piecewise linear membership functions $\mu_{\tilde{G}_l}$, $l = 1, 2, \ldots, q$:

$$\mu_{\tilde{G}_l}(y) = \begin{cases} 0 & \text{if } y > f_l^0 \\ \dfrac{y - f_l^0}{f_l^1 - f_l^0} & \text{if } f_l^1 \leq y \leq f_l^0 \\ 1 & \text{if } y < f_l^1, \end{cases} \tag{7}$$

where f_l^0 and f_l^1 are parameter values determined by a DM. Figure 3 shows the linear membership functions of fuzzy goal $\tilde{G}_l$.

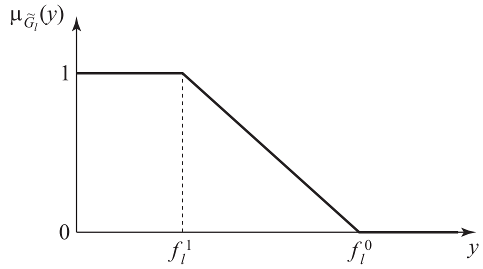

Fig. 3. Linear membership function $\mu_{\tilde{G}_l}$ of a fuzzy goal $\tilde{G}_l$

One of reasonable solving processes is to maximize the degree of possibility or necessity that the fuzzy goals $\tilde{G}_l$ are attained. When formulated problems involve only fuzziness, possibilistic programming [7], [8] is applicable to solving them.

Unfortunately, however, possibilistic programming approaches cannot directly be applied to solving mathematical programing problems with discrete fuzzy random variables. This is because the degrees of possibility or necessity are not constants but vary dependent on events ω_{lk}.

Now we shall discuss how to construct optimization criteria in order to solve the problems involving both fuzziness and randomness. Assuming that a certain event ω_{lk} has occurred, on the basis of possibility theory [3], the degree of possibility that $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$ satisfies fuzzy goal $\tilde{G}$ (namely, the degree of possibility that the objective function value $\tilde{\tilde{\boldsymbol{C}}}_{lk}\boldsymbol{x}$, $l = 1, 2, \ldots, q$, $k = 1, 2, \ldots, r_l$ for any event $\omega_{lk} \in \Omega_l$ is substantially less than or equal to a certain aspiration level f_l) is defined as

$$\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \;\triangleq\; \sup_y \min\left\{\mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}(y),\ \mu_{\tilde{G}_l}(y)\right\}. \tag{8}$$

Figure 4 illustrates the degree of possibility defined by (8) for a fixed event ω_{lk}, which is the ordinate of the crossing point between the membership functions of fuzzy goal $\tilde{G}_l$ and the objective function $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$.

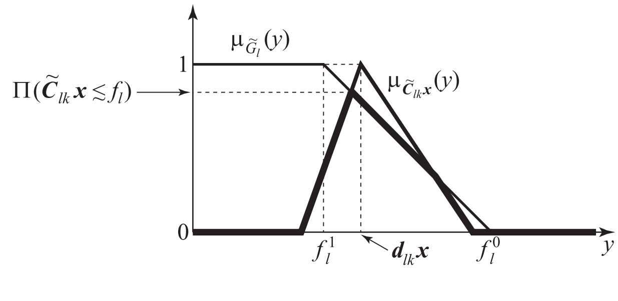

Fig. 4. Degree of possibility $\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right)$.

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The possibility measure is recommended to *optimistic* DMs. Since DMs are not always optimistic in general, we introduce an optimization criterion based on necessity measures in order to construct an optimization criterion for *pessimistic* DMs.

On the basis of possibility theory [3], the degree of necessity that the objective function value $\tilde{\bar{\boldsymbol{C}}}_{lk}\boldsymbol{x}$, $l = 1, 2, \ldots, q$, $k = 1, 2, \ldots, r_l$ for any event $\omega_{lk} \in \Omega_l$ satisfies the fuzzy goal $\tilde{G}_l$ is defined as

$$N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \triangleq \inf_y \max\left\{1 - \mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}(y),\ \ \mu_{\tilde{G}_l}(y)\right\}. \tag{9}$$

Figure 5 illustrates the degrees of necessity defined by (9), which is the ordinate of the crossing point between the membership functions of fuzzy goal $\tilde{G}_l$ and the upside-down of the membership function of the objective function $\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}$.

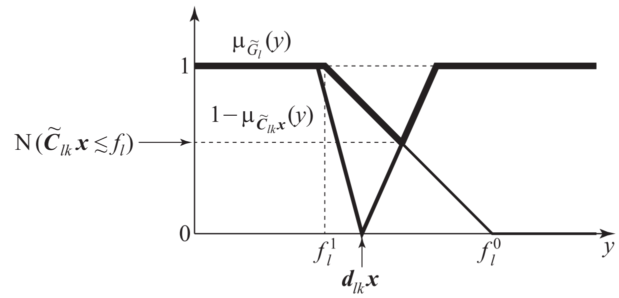

Fig. 5. Degree of necessity $N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right)$.

By using p_{lk} which is the probability that an event w_{lk} occurs, the optimization criterion, the probabilistic expectations of possibility and necessity are defined as follows:

$$E\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \triangleq \sum_{k=1}^{r_l} p_{lk} \cdot \Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \tag{10}$$

$$E\left[N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \triangleq \sum_{k=1}^{r_l} p_{lk} \cdot N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \tag{11}$$

where $E[\cdot]$ denotes a probabilistic expectation.

Optimization criteria defined as (10) and (11) take into consideration both fuzziness and randomness involved in the coefficients of the mathematical programming problem. Some previous papers [11], [17] addressed optimization problems based on these criteria. A variance minimization model for linear programming problems with discrete fuzzy random variables [12] was also proposed, but those with downside risk measures have not been considered so far.

V. Possibilistic LPM models for downside risk-aversion

In this section, we propose a new downside risk measure in fuzzy stochastic decision making environments, called *possibilistic low partial moment* (PLPM), which is an extended concept of classical low partial moment (LPM) through the incorporation of possibility theory into the LPM. As PLPM, we propose two measures, namely, *possibility low partial moment* and *necessity low partial moment*.

Firstly, we define the possibility low partial moment as follows:

$$LPM_\gamma\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \triangleq \sum_{k=1}^{r_l} p_{lk}\cdot\left|\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\pi}_l\right|_-^\gamma, \tag{12}$$

where $|a|_- \triangleq \max\{-a, 0\}$, and γ is a positive integer which represents the degrees of moment. When $\gamma = 1$, (12) means the semi-absolute deviation of the possibility $\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)$. On the other hand, when $\gamma = 2$, (12) represents the semi-variance of the possibility.

Then, we consider the following optimization problem which minimizes the possibility lower partial moment under the constraint of probabilistic possibility expectation:

$$\left.\begin{array}{ll}\text{minimize} & LPM_\gamma\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right],\ l = 1, 2, \ldots, q \\ \text{subject to} & E\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \geq \hat{\pi}_l^E,\ \ l = 1, 2, \ldots, q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{13}$$

Here, we focus on the following minimax problem:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\ldots,q} LPM_\gamma\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \\ \text{subject to} & E\left[\Pi\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \geq \hat{\pi}_l^E,\ \ l = 1, 2, \ldots, q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{14}$$

or equivalently

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\ldots,q}\left\{\sum_{k=1}^{r_l} p_{lk}\cdot\left|\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\pi}_l^G\right|_-^\gamma\right\} \\ \text{subject to} & \displaystyle\sum_{k=1}^{r_l} p_{lk}\cdot\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \geq \hat{\pi}_l^E, \\ & \qquad\qquad l = 1, 2, \ldots, q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0},\end{array}\right\} \tag{15}$$

where $\hat{\pi}_l^G$ is a target value of the possibility, and $\hat{\pi}_l^E$ is a satisficing value of probabilistic expectation of possibility. All the values of $\hat{\pi}_l^G$ and $\hat{\pi}_l^E$ are given by decision makers as positive constants greater than or equal to 0 and less than or equal to 1.

Now we consider the case where f_l^1 and f_l^0 are determined by the following calculation:

$$\left.\begin{array}{rcl}f_l^1 & = & \displaystyle\min_k \min_{\boldsymbol{x}\in X} \sum_{j=1}^{n}\sum_{k=1}^{r_l} p_{lk} d_{ljk} x_j \\ f_l^0 & = & \displaystyle\max_k \max_{\boldsymbol{x}\in X} \sum_{j=1}^{n}\sum_{k=1}^{r_l} p_{lk} d_{ljk} x_j.\end{array}\right\} \tag{16}$$

Assuming that β_{ijk} is not dependent on k and that β_{ijk} is replaced by β_{ij}, $\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right)$ is calculated as follows [12]:

$$\begin{array}{rcl}\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) & \triangleq & \displaystyle\sup_y \min\left\{\mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}(y),\ \mu_{\tilde{G}_l}(y)\right\} \\ & = & \dfrac{\displaystyle\sum_{j=1}^{n}(\beta_{lj} - d_{ljk})x_j + f_l^0}{\displaystyle\sum_{j=1}^{n}\beta_{lj}x_j - f_l^1 + f_l^0}.\end{array} \tag{17}$$

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By introducing z_{lk}^{Π}, $l = 1,2,\dots,q$, $k = 1,2,\dots,r_l$, $\left|\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\pi}_l^G\right|_-$ is represented as follows:

$$\begin{aligned}&\left|\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\pi}_l^G\right|_- \\ &\quad = \min\left\{z_{lk}^{\Pi}\,\middle|\,\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) + z_{lk}^{\Pi} \geq \hat{\pi}_l^G\right\}.\end{aligned}$$

Consequently, problem (15) is equivalently transformed into the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\dots,q}\left\{\sum_{k=1}^{r_l} p_{lk}\cdot\left(z_{lk}^{\Pi}\right)^{\gamma}\right\} \\ \text{subject to} & \Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) + z_{lk}^{\Pi} \geq \hat{\pi}_l^G,\ z_{lk}^{\Pi} \geq 0, \\ & \quad l = 1,2,\dots,q, k = 1,\dots,r_l \\ & \displaystyle\sum_{k=1}^{r_l} p_{lk}\cdot\Pi\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) \geq \hat{\pi}_l^E, \\ & \quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{18}$$

By using (17), problem (18) is transformed into the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\dots,q}\left\{\sum_{k=1}^{r_l} p_{lk}\cdot\left(z_{lk}^{\Pi}\right)^{\gamma}\right\} \\ \text{subject to} & \displaystyle\sum_{j=1}^{n}\beta_{lj}z_{lk}^{\Pi}x_j + \sum_{j=1}^{n}\left(\beta_{lj} - d_{ljk} - \hat{\pi}_l^G\right)x_j \\ & \quad +(f_l^0 - f_l^1)z_{lk}^{\Pi} \geq (f_l^0 - f_l^1)\hat{\pi}_l^G - f_l^0, \\ & z_{lk}^{\Pi} \geq 0,\ l = 1,2,\dots,q, k = 1,\dots,r_l \\ & \displaystyle\sum_{j=1}^{n}\left(\beta_{lj} - \hat{\pi}_l^E - \sum_{k=1}^{r_l} p_{lk}\cdot d_{ljk}\right)x_j \\ & \quad \geq (f_l^0 - f_l^1)\hat{\pi}_l^E - f_l^0, \\ & \quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\end{array}\right\} \tag{19}$$

or equivalently

$$\left.\begin{array}{ll}\text{minimize} & \theta \\ \text{subject to} & \displaystyle\sum_{k=1}^{r_l} p_{lk}\cdot\left(z_{lk}^{\Pi}\right)^{\gamma} \leq \theta,\ l = 1,2,\dots,q \\ & \displaystyle\sum_{j=1}^{n}\beta_{lj}z_{lk}^{\Pi}x_j + \sum_{j=1}^{n}\left(\beta_{lj} - d_{ljk} - \hat{\pi}_l^G\right)x_j \\ & \quad +(f_l^0 - f_l^1)z_{lk}^{\Pi} \geq (f_l^0 - f_l^1)\hat{\pi}_l^G - f_l^0, \\ & z_{lk}^{\Pi} \geq 0,\ l = 1,2,\dots,q, k = 1,\dots,r_l \\ & \displaystyle\sum_{j=1}^{n}\left(\beta_{lj} - \hat{\pi}_l^E - \sum_{k=1}^{r_l} p_{lk}\cdot d_{ljk}\right)x_j \\ & \quad \geq (f_l^0 - f_l^1)\hat{\pi}_l^E - f_l^0, \\ & \quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{20}$$

It should be noted here that problem (20) is a polynomial optimization problem [26], which can be solved by using semidefinite programming relaxation under some conditions.

A. Necessity lower partial moment model

In a manner similar to the possibility low partial moment, we propose a necessity low partial moment as follows:

$$LPM_{\lambda}\left[N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \triangleq \sum_{k=1}^{r_l} p_{lk}\cdot\left|N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\nu}_l\right|_-^{\lambda}. \tag{21}$$

In order to consider the case where a decision maker is pessimistic, we propose the following minimax problem which minimizes the maximum necessity low partial moment under the constraint of the probabilistic necessity expectation:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\dots,q} LPM_{\gamma}\left[N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \\ \text{subject to} & E\left[N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right)\right] \geq \hat{\nu}_l^E,\quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\end{array}\right\} \tag{22}$$

or equivalently

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\dots,q}\left\{\sum_{k=1}^{r_l} p_{lk}\cdot\left|N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right) - \hat{\nu}_l^G\right|^{\lambda}\right\} \\ \text{subject to} & \displaystyle\sum_{k=1}^{r_l} p_{lk}\cdot N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right) \geq \hat{\nu}_l^E, \\ & \quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{23}$$

Assume that f_l^0 and f_l^0 are determined by (16), and that γ_{ijk} is replaced by γ_{ij} which does not depend on k. Then, $N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right)$ is calculated as the following linear fractional function of decision variables x_j [12]:

$$\begin{aligned}N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) &\triangleq \inf_y \max\left\{1 - \mu_{\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x}}(y),\ \ \mu_{\tilde{G}_l}(y)\right\} \\ &= \frac{-\displaystyle\sum_{j=1}^{n} d_{ljk}x_j + f_l^0}{\displaystyle\sum_{j=1}^{n}\gamma_{lj}x_j - f_l^1 + f_l^0}.\end{aligned} \tag{24}$$

By introducing z_{lk}^{N}, $l = 1,2,\dots,q$, $k = 1,2,\dots,r_l$, $\left|N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\nu}_l^G\right|_-$ is represented as follows:

$$\begin{aligned}&\left|N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) - \hat{\nu}_l^G\right|_- \\ &\quad = \min\left\{z_{lk}^{N}\,\middle|\,N\left(\tilde{\boldsymbol{C}}_{lk}\boldsymbol{x} \lesssim f_l\right) + z_{lk}^{N} \geq \hat{\nu}_l^G\right\}.\end{aligned}$$

Consequently, problem (23) is equivalently transformed into the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \displaystyle\max_{l=1,2,\dots,q}\left\{\sum_{k=1}^{r_l} p_{lk}\cdot\left(z_{lk}^{N}\right)^{\lambda}\right\} \\ \text{subject to} & N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right) + z_{lk}^{N} \geq \hat{\nu}_l^G,\ z_{lk}^{N} \geq 0, \\ & \quad l = 1,2,\dots,q, k = 1,\dots,r_l \\ & \displaystyle\sum_{k=1}^{r_l} p_{lk}\cdot N\left(\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x} \lesssim f_l\right) \geq \hat{\nu}_l^E, \\ & \quad l = 1,2,\dots,q \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.\end{array}\right\} \tag{25}$$

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In a manner similar to the transformation of problem (18) into (20), it is easily shown that problem (25) is transformed into a polynomial optimization problem, which can be solved by using semidefinite programming relaxation under some conditions.

VI. Conclusion

In this paper, we have considered linear optimization (linear programming) problems with discrete fuzzy random variables. To take into consideration downside risk in fuzzy stochastic environments, we have proposed new models based on low partial moment using possibility and necessity measures, which is a generalized version of semi-variance and semi-absolute deviation of possibility and necessity. It has been shown that the problem can be transformed into polynomial optimization problems, which can be solved by semidefinite programming relaxation under some conditions. The framework proposed in this paper can be applied to solving a wide variety of real-world decision making problems which can be modeled as linear programming problems.

Now we are constructing other downside risk measures based on (conditional) value-at-risk using possibility and necessity measures. We will present the results somewhere in the near future.

References

[1] Aiche, F., Abbas, M., & Dubois, D. (2013). Chance-constrained programming with fuzzy stochastic coefficients. *Fuzzy Optimization Decision Making*, 12(2), 125–152.

[2] Birge, J.R. & Louveaux, F. (2011). *Introduction to Stochastic Programming*, Second Edition. Springer.

[3] Dubois, D., & Prade, H. (1988). *Possibility theory*. John Wiley & Sons.

[4] Gil, M.A. (2004). Fuzzy random variables: development and state of the art. In *Mathematics of Fuzzy Systems, Linz Seminar on Fuzzy Set Theory* (11–15).

[5] Gil, M.A., Lopez-Diaz, M., & Ralescu, D.A. (2006). Overview on the development of fuzzy random variables. *Fuzzy Sets and Systems*, 157(19), 2546–2557.

[6] Giri, P.K., Maiti, M.K., & Maiti, M. (2014). Entropy based solid transportation problems with discounted unit costs under fuzzy random environment. *OPSEARCH*, 51(4), 479–532.

[7] Inuiguchi, M. & Ichihasi, H. (1990). Relative modalities and their use in possibilistic linear programming. *Fuzzy Sets and Systems*, 35(3), 303–323.

[8] Inuiguchi, M. & Ramik, J. (2000). Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems* 111(1), 3–28.

[9] Katagiri, H., & Ishii, H. (2000). Chance constrained bottleneck spanning tree problem with fuzzy random edge costs. *Journal of the Operations Research Society of Japan*, 43(1), 128–137.

[10] Katagiri, H., Kato, K. & Hasuike, T. (2012). A random fuzzy minimum spanning tree problem through a possibility-based value at risk model, *Expert Systems with Applications*, 39(12), 205–220.

[11] Katagiri, H., Mermri, E.B., Sakawa, M., Kato, K., & Nishizaki, I. (2005). A possibilistic and stochastic programming approach to fuzzy random MST problems, *IEICE Transactions on Information and Systems*, 88(8), 1912–1919.

[12] Katagiri, H., & Sakawa, M. (2003). A study on fuzzy random linear programming problems based on possibility and necessity measures. In *Fuzzy Sets and Systems-IFSA 2003* (725–732). Springer.

[13] Katagiri, H. & Sakawa, M. (2011). Interactive multiobjective fuzzy random programming through the level set-based probability model. *Information Sciences*, 181(9), 1641–1650.

[14] Katagiri, H., Sakawa, S. & Ishii, H. (2004). Fuzzy random bottleneck spanning tree problems using possibility and necessity measures, *European Journal of Operational Research* 152(1), 88–95.

[15] Katagiri, H., Sakawa, M., & Ishii, H. (2005). Studies of stochastic programming models using possibility and necessity measures for linear programming problems with fuzzy random variable coefficients. *Electronics and Communications in Japan (Part III: Fundamental Electronic Science)*, 88(1), 68–75.

[16] Katagiri, H., Sakawa, M., & Ishii, H. (2005). A study on fuzzy random portfolio selection problems based on possibility and necessity measures. *Scientiae Mathematicae Japonicae*, 61(2), 361–370.

[17] Katagiri, H., Sakawa, M., Kato, K. & Nishizaki, I. (2004). A fuzzy random multiobjective 0-1 programming based on the expectation optimization model using possibility and necessity measures. *Mathematical and Computer Modelling*, 40(3–4), 411–421.

[18] Katagiri, H., Sakawa, M., Kato, K. & Nishizaki, I. (2008). Interactive multiobjective fuzzy random linear programming: maximization of possibility and probability. *European Journal of Operational Research*, 188(2), 530–539.

[19] Katagiri, H., Sakawa, M., Kato, K., & Ohsaki, S. (2005). An interactive fuzzy satisficing method based on the fractile optimization model using possibility and necessity measures for a fuzzy random multiobjective linear programming problem. *Electronics and Communications in Japan (Part III: Fundamental Electronic Science)*, 88(5), 20–28.

[20] Katagiri, H., Uno, T., Kato, K., Tsuda, H. & Tsubaki, H. (2013). Random fuzzy multiobjective programming: Optimization of possibilistic Value at Risk (pVaR), *Expert Systems with Applications* 40(2), 563–574.

[21] Kaufmann, A., & Gupta, M.M. (1985). Introduction to Fuzzy Arithmetic. Theory and Applications, Van Nostrand Reinhold, New York (1985).

[22] Klement, E.P., Puri, M.L. & Ralescu, D.A. (1986). Limit theorems for fuzzy random variables. *Proceedings of the Royal Society of London* A 407, 171–182.

[23] Kruse, R. & Meyer, K.D. (1987). *Statistics with Vague Data*. D. Reidel Publishing Company.

[24] Kwakernaak, H. (1978). Fuzzy random variables-1. Definitions and theorems. *Information Sciences*, 15(1), 1–29.

[25] Kwakernaak, H. (1979). Fuzzy random variables-II. Algorithms and examples for the discrete case. *Information Sciences*, 17(3), 253–278.

[26] Lasserre, J. B. (2001). Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3), 796-817.

[27] Liu, B. (2001). Fuzzy random chance-constrained programming. *IEEE Transaction on Fuzzy Systems*, 9(5), 713–720.

[28] Liu, Y.-K., Liu, B. (2003). A class of fuzzy random optimization: expected value models. *Information Sciences*, 155(1–2), 89–102.

[29] Luhandjula, M.K. (1996). Fuzziness and randomness in an optimization framework. *Fuzzy Sets and Systems*, 77(3), 291–297.

[30] Luhandjula, M.K. (2006). Fuzzy stochastic linear programming: survey and future research directions. *European Journal of Operational Research*, 174(3), 1353–1367.

[31] Luhandjula, M.K. & Gupta, M.M. (1996). On fuzzy stochastic optimization. *Fuzzy Sets and Systems*, 81(1), 47–55.

[32] Nematian, J. (2014). A robust single row facility layout problem with fuzzy random variables. *The International Journal of Advanced Manufacturing Technology*, 72(1-4), 255–267.

[33] Puri, M.L. & Ralescu, D.A. (1986). Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, 14(2), 409–422.

[34] Shapiro, A.F. (2009). Fuzzy random variables, *Insurance: Mathematics and Economics*, 44(2), 307–314.

[35] Yano, H. (2014). Fuzzy decision making for fuzzy random multiobjective linear programming problems with variance covariance matrices. *Information Sciences*, 272, 111–125.

[36] Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353.

[37] Zadeh, L.A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1), 3–28.

[38] Zimmermann, H.-J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets and Systems*, 1(1), 5–55.

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