Numerical Approximation of ARL on Modified EWMA Control Chart for MA(1) Process

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Abstract— **This article is to evaluate the average run length (ARL) by using the numerical integral equation (NIE) via Gauss-Legendre quadrature rules on modified exponentially weighted moving average (Modified EWMA) control chart when the observation are the first order moving average MA(1) with exponential white noise. Additionally, this is also extended to compare efficiency of Modified EWMA with EWMA procedure. The performance of Modified EWMA chart was found to be superior to EWMA procedure for all magnitudes of change when smoothing parameter (λ) is greater than or equal to 0.075.**

Index Terms— **Modified EWMA, Moving Average Process, Average Run Length, Numerical Integral Equation**

I. Introduction

Several methods are widely used in order to improve capability through the reduction of variability. One of the methods is called statistical process control (SPC). It is extremely effective for monitoring and detecting the change of process in the control chart. It is also popular and actually provides an advantage in a variety of applications. The first control chart used in widely statistical control process was presented by Shewhart. Shewhart control chart was useful for detecting the large shift in process. When the assumptions are independent and have normal distribution.

There are many control charts used in order to monitor either fields of their applications. One of the extensive benefits of using a control chart is the exponentially weighted moving average (EWMA) as proposed by Roberts [1]. It is widely used to detect or monitor small changes in a process. Neubauer [2], Psarakis and Papaleonida [3] and Mawonike and Nkomo [4] showed that this is a good alternative to the Shewhart control chart and much more effective than the original procedure when monitoring small changes and auto correlated data processes or free form distribution. Diversity of the EWMA procedure have been developed by researchers such as Montgomery and Mastrangelo [5], Alwan and Roberts [6], Lu and Reynolds [7].

Literature concerning the advantage of using EWMA chart was conducted by Barbeito et al [8], Serel [9], Yang [10], Serel and Moskowitz [11].

Recently, Patel and Divecha [12] initially presented the modified exponentially weighted moving average (modified EWMA) control chart which is very effective when detecting small and abrupt shifts in the monitoring process. Moreover, this also had good performance for observations that are autocorrelated.

There are several situations in which control charts can be compared by average run length which is a traditional measurement of the control chart capability denoted by ARL. The average number of point or observation plotted within the control limit by the process is an in-control process until there exists a point or observation false signal out of the control limit denoted by ARL_0. The average number of points or observations plotted within the control limit by the process is an out-of-control process until there is a point or observation plotted false signal denoted by ARL_1. ARL_0 is large enough to maintain a level of false alarm at an acceptable level, it will be accepted out the control limit. Completely, ARL_1 should be as small as possible. Furthermore, the are many methods that can evaluate the ARL such as explicit formulas, Markov Chain approach (MCA), Monte Carlo simulation (MC), Martingale approach and numerical integral equation (NIE).

Champ and Ridgon [13] studied CUSUM and EWMA charts using the Markov chain and integral equation approaches to evaluate the ARL. Later, Mastrangelo and Montgomery [14] evaluated the performance of EWMA control charts for serially-correlated processes by using the Monte Carlo simulation technique. Areepong [15] derived the result when evaluating ARL_0 and ARL_1 for EWMA, CUSUM and Shiryayev-Roberts charts using the explicit formulas, MCA, NIE and MC simulation methods. Mititelu et al. [16] used the explicit formulas to find the ARL by Fredholm type Integral Equation for a EWMA control chart with Laplace distribution and CUSUM control charts with Hyperexponential distribution. Suriyakat et al. [17] used the Integral Equation method to solve the ARL and derived explicit expression for $AR(1)$ procedure on EWMA control chart processes with exponential white noise. Recently, Suntornwat et al. [18] proposed a solution for ARL which evaluated the integral equation technique on the EWMA control chart for ARIMA process and also compared the analytical solution.

Manuscript received October 5, 2018; revised November 28, 2018. This work was supported in part by graduate college, King Mongkut's University of Technology North Bangkok, 10800, Thailand.

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The objective of this study is to propose a numerical integral equation of ARL on a modified EWMA control chart for the $MA(1)$ model with exponential white noise. Moreover, this will be extended to compare the effective performance with EWMA control charts. The collocation of this study is as follows: In the second part, the methodology of approximation for ARL on modified EWMA control chart is proposed. Numerical results are presented in the third part. The conclusion is discussed in the fourth part. Finally, references are provided in the fifth part.

II. Modified EWMA Control Chart and Model for Observation

The EWMA control chart usually carries on by monitoring and detecting small changes in process mean. It was originally proposed by Robert (1959). The EWMA control chart can be expressed by the recursive equation below

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t \ , \ t = 1,2,3,... \qquad (1)$$

where λ is an exponential smoothing parameter and $0 < \lambda \le 1$ start with value $Z_0 = X_0 = \mu_0$ and μ_0 is target mean. Observations X_t are independent random variables with variance σ^2, the upper control limit (UCL) and lower control limit (LCL) are

$$\mu_0 \pm L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \qquad (2)$$

where L which is suitable to control width limit. In the case of standard normal distribution, L is usually set as 3.

A modified EWMA was developed and presented by Patel and Divecha (2011). It is very effective in detecting small shifts when monitoring processes and free form of distribution.

The modified EWMA chart is defined by the recursive equation below

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t + (X_t - X_{t-1}) \ , \ t = 1,2,3,... \qquad (3)$$

where X_t is an observation which is a sequence of moving average order 1 ($MA(1)$) process, $Z_0 = u$ and $X_0 = v$ are initial values and λ is an exponential smoothing parameter and $0 < \lambda \le 1$. The upper control limit (UCL) and lower control limit (LCL) of the Modified EWMA chart are

$$\mu_0 \pm L\sigma\sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}} \qquad (4)$$

where μ_0 is the target mean, σ^2 is the process variance and L is suitable in control width limit.

The $MA(1)$ process can be described by the following recursion

$$X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1} \qquad (5)$$

where ε_t is a white noise process assumed to have exponential distribution, θ is a moving average coefficient which $|\theta| \le 1$ or $-1 \le \theta \le 1$ and given the initial value of $\varepsilon_0 = s$.

Stopping time of the modified EWMA chart is given by

$$\tau_b = \inf\{t > 0; Z_t > b\}, b > u \qquad (6)$$

where τ_b is the stopping time, b is the upper control limit (UCL).

III. Approximation For ARL on Modified EWMA Chart

Let $L(u)$ denote by the ARL for $MA(1)$ process. To define function $L(u)$ following

$$ARL = L(u) = \mathbf{E}_\infty(\tau_b) \qquad (7)$$

where $\mathbf{E}_\infty$ is the expectation.

According to a method similar to Crowder [19] and VanBrackle and Reynold [20], the formula for $L(u)$ can be written as follows

$$L(u) = 1 + \frac{1}{\lambda+1}\int_0^b L(k) f\left(\frac{k-(1-\lambda)u+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right)dk \qquad (8)$$

The numerical method to solve the integral equation uses the quadrature rule approach which approximates the integral by finite sum of areas of rectangles with base b/m with heights chosen as the values of f at midpoints of the one-sided interval. This divides $[0,b]$ into a partition $0 \le a_1 \le a_2 \le ... \le a_m \le b$ and a set of constant weights $w_j = b/m \ge 0$. The approximation for an integral can be expressed by

$$\int_0^b \mathrm{L(k)\,f(k)\,dk} \approx \sum_{j=1}^m w_j f(a_j) \qquad (9)$$

The integral equation (8) can be approximated by

$$L(a_i) = 1 + \frac{1}{\lambda+1}\sum_{j=1}^m w_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right)$$

$; \ i = 1,2,3,...,m$

It can be written in a matrix form as below

$$L_{m\times1} = 1_{m\times1} + R_{m\times m}L_{m\times1} \text{ or } L_{m\times1} = (I_m - R_{m\times m})^{-1}1_{m\times1} \qquad (10)$$

An approximation of the NIE method for function $L(u)$ is

$$\tilde{L}(u) = 1 + \frac{1}{\lambda+1}\sum_{j=1}^m w_j L(a_j) f\left(\frac{a_j-(1-\lambda)u+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) \qquad (11)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}$; $j = 1,2,3,...,m$.

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

We implemented the numerical procedure given by equation (11) for the first order moving average with exponential white noise when the number of divisions (*m*) is 1,000. The evaluations of average run length are shown in Table I. The performance comparison for in control process (ARL_0) of Modified EWMA against EWMA control chart is presented in Table II - IV. The approximation of function $\tilde{L}(u)$ on a EWMA control chart was evaluated by Petcharat [21].

IV. Numerical Results

In this section, the *ARL* on modified EWMA chart by using Gauss-Legendre numerical integral equation when observations are the first order moving average process with exponential white noise is proposed, and then performance of the Modified EWMA and EWMA control charts will be compared.

TABLE I
ARL_0 OF MODIFIED EWMA FOR MA(1) PROCESS WHEN GIVEN $u=1, \mu=2$.

λ	θ	b	ARL_0	CPU time (second)
0.03	0.2	0.4499254	370.0289	19.095
	-0.2	0.3008257	370.0279	19.033
0.05	0.2	0.4522293	370.0229	18.970
	-0.2	0.3019502	370.0198	18.939
0.075	0.2	0.4552539	370.0407	19.141
	-0.2	0.3034602	370.0362	19.187
0.08	0.2	0.4558771	370.0155	19.001
	-0.2	0.3037753	370.0128	19.251
0.10	0.2	0.4584297	370.0244	19.157
	-0.2	0.3050782	370.0280	19.016
0.12	0.2	0.4610734	370.0205	19.250
	-0.2	0.3064454	370.0223	19.094

In Table I, the average run length is evaluated by using (11) on modified EWMA control chart when $\beta=1$, $\mu=2$, $\theta=\pm 0.2$, $\lambda=$ 0.03, 0.05, 0.075, 0.08, 0.10, 0.12 for $ARL_0=370$.

TABLE II
COMPARISON OF *ARL* BETWEEN EWMA AND MODIFIED EWMA CHARTS FOR MA(1) PROCESS WHEN GIVEN $u=1, \mu=2, ARL_0=370$ AND $\lambda=0.03, 0.05$.

Shift size (δ)	$\lambda=0.03$		$\lambda=0.05$	
	EWMA	MoEWMA	EWMA	MoEWMA
0.00	370.0241 (16.660)*	370.0289 (19.095)	370.0211 (16.568)	370.0229 (18.970)
0.005	316.4396 (16.879)	141.4034 (19.266)	338.0712 (16.520)	138.1735 (19.235)
0.01	271.0485 (16.661)	87.3909 (19.141)	309.1561 (16.536)	84.9555 (19.281)
0.03	148.2281 (16.505)	34.5734 (19.001)	218.0927 (16.676)	33.4702 (19.173)
0.05	83.1136 (16.692)	21.5683 (19.173)	155.9366 (16.723)	20.8757 (19.126)
0.07	47.7912 (16.660)	15.6938 (19.125)	112.9399 (16.786)	15.1970 (19.250)
0.10	21.9009 (16.786)	11.1723 (19.313)	71.2393 (16.645)	10.8302 (19.437)
0.30	1.2472 (16.630)	4.0336 (19.313)	6.3616 (16.521)	3.9395 (19.188)
0.50	1.0093 (16.879)	2.6584 (19.188)	1.7962 (16.786)	2.6107 (19.111)
1.00	1.0000 (16.942)	1.7094 (19.406)	1.0340 (16.802)	1.6916 (19.359)

* The parentheses are CPU times in seconds.

TABLE III
COMPARISON OF *ARL* BETWEEN EWMA AND MODIFIED EWMA CHARTS FOR MA(1) PROCESS WHEN GIVEN $u=1, \mu=2, ARL_0=370$ AND $\lambda=0.075, 0.08$.

Shift size (δ)	$\lambda=0.075$		$\lambda=0.08$	
	EWMA	MoEWMA	EWMA	MoEWMA
0.00	370.0395 (16.693)	370.0407 (19.141)	370.0216 (16.786)	370.0155 (19.001)
0.005	349.4666 (16.661)	134.3564 (19.079)	350.9116 (16.598)	133.6167 (19.142)
0.01	330.2194 (16.895)	82.1097 (19.250)	332.9577 (16.754)	81.5638 (19.266)
0.03	264.6783 (16.786)	32.1948 (19.266)	271.2075 (16.723)	31.9522 (18.986)
0.05	213.8999 (16.692)	20.0769 (19.188)	222.5932 (16.598)	19.9252 (19.064)
0.07	174.2186 (16.692)	14.6244 (19.547)	184.0124 (16.801)	14.5157 (19.047)
0.10	129.8412 (16.739)	10.4359 (19.251)	140.0768 (16.630)	10.3611 (19.094)
0.30	26.0075 (16.802)	3.8309 (19.344)	31.3645 (16.770)	3.8103 (19.048)
0.50	8.3619 (17.082)	2.5556 (19.376)	10.7440 (16.661)	2.5451 (19.141)
1.00	1.9550 (16.801)	1.6710 (19.469)	2.4539 (17.051)	1.6671 (19.360)

TABLE IV
COMPARISON OF *ARL* BETWEEN EWMA AND MODIFIED EWMA CHARTS FOR MA(1) PROCESS WHEN GIVEN $u=1, \mu=2, ARL_0=370$ AND $\lambda=0.10, 0.12$.

Shift size (δ)	$\lambda=0.10$		$\lambda=0.12$	
	EWMA	MoEWMA	EWMA	MoEWMA
0.00	370.0203 (17.426)	370.0244 (19.157)	370.0246 (16.911)	370.0205 (19.250)
0.005	355.5932 (16.724)	130.7603 (19.110)	360.6031 (16.910)	128.0368 (19.094)
0.01	341.8563 (16.785)	79.4629 (19.094)	351.5088 (16.677)	77.4787 (19.220)
0.03	293.0758 (16.802)	31.0224 (19.313)	318.1319 (17.004)	30.1515 (19.437)
0.05	252.6590 (17.020)	19.3443 (19.048)	288.9824 (17.035)	18.8012 (19.453)
0.07	218.9666 (16.692)	14.0997 (19.188)	263.4097 (16.786)	13.7110 (19.375)
0.10	178.3091(16.739)	10.0748 (19.172)	230.6131 (16.910)	9.8073 (19.579)
0.30	57.2199 (16.692)	3.7312 (19.235)	110.0205 (16.598)	3.6572 (19.344)
0.50	24.6987 (17.145)	2.5048 (19.220)	62.6745 (16.739)	2.4671 (19.672)
1.00	6.4918 (17.176)	1.6520 (19.390)	23.3963 (17.192)	1.6378 (19.453)

Tables II – IV are used to compare approximate *ARL* of EWMA and modified EWMA control chart given $\mu=2$, $\theta=0.2$, shift size value (δ) = 0.00, 0.005, 0.01, 0.03, 0.05, 0.07, 0.10, 0.30, 0.50, 1.00 for ARL_0=370 for λ = 0.03, 0.05, 0.75, 0.08, 0.10 and 0.12. The result appeared to be when λ = 0.03 and 0.05, performance of the modified EWMA is better than EWMA control chart for δ which is less than or equal to 0.10. However, when λ = 0.75, 0.08, 0.10 and 0.12, performance of the Modified EWMA control chart is better than the EWMA control chart for all magnitudes of change of the process mean.

V. Conclusion

The numerical integral equation of the average run length of modified EWMA control chart for the observations of the first order moving average with exponential white noise was presented. It was shown that recommended procedure is considerably closer to the computational time of the latter scheme. In particular, it detected the small shift better than another charts. When the exponential smoothing parameters were given by 0.075, 0.08, 0.10 and 0.12, the monitoring of all magnitudes of Modified EWMA chart performed better than EWMA chart. It was also discovered that either suggested scheme performance depends on the optimal smoothing parameter.

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ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)