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Regulated Functions with values in the Banach Algebra of Quaternions

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Abstract—**In this paper we deal with the notion of regulated functions with values in a Banach algebra $\mathcal{A}$, we prove some results and present examples using quaternions. The physical meanning of this kind of functions is introduced and some particular results are presented. We consider then, the Dushnik integral for these functions and we construct a correspondent linear integral functional on the Banach algebra of all regulated functions $G([a,b],\mathcal{A})$.**

Keywords: ***Regulated function, Banach Algebras, Quaternions.***

1 Introduction

Sometimes to describe physical events we need a model that has, besides the basic operations of linear spaces and the notion of size of their elements, an internal multiplication completely compatible with the normed linear space structure. These spaces are known as Banach algebras, subject that was treated by J. von Neumann, I. M. Gelfand and M. A. Naimark, among others, in the years 1930-60. For details see ([2]) Our interest here is to study the set of all well-behaved funtions $f : I = [a,b] \subset \mathbb{R} \to \mathcal{A}$, i. e., the set of all functions that have the lateral limits $f(t-)$ and $f(t+)$, known as regulated functions, for every $t \in]a,b[$ ($f(b-) = f(b)$ and $f(a+) = f(a)$) when $\mathcal{A}$ is a Banach algebra. The definition of regulated function first appears in Dieudonne's book [1]. The space of regulated functions was approached by several authors in the last years, see for example, ([3] [4] [8]) The classical notation for this set of functions is $G([a,b],\mathcal{A})$ and it is a Banach space with the uniform convergence norm. In Section 2 we present the notions of regulated functions, Dushnik integral and Banach algebras, and we present proofs of some results to guarantee that $G([a,b],\mathcal{A})$ inherits the structure of A, in other words, it is also a Banach algebra. Section 3 describes the set of Quaternions and recalls how the elements of this set can be used to represent rotations of bodies in three dimensions. Finally, in Section 4, the notions and results are then applied in the special case when $I = [0,T]$ and $\mathcal{A} = Quat$, the set of quaternions. This set has the structure of non-commutative Banach algebra and has been used in graphic computation and modeling 3D rotations, and we will discuss the behavior of functions with values in Banach algebra.

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2 Regulated Functions Banach Algebra Valued

Roughly speaking, algebras are simultaneously normed linear spaces and rings. They are structures with an addition, a scalar multiplication, an internal multiplication and a norm, all completly compatible. Formally we have that

Definition 1 *A Banach algebra A over the complex number field $\mathbb{C}$ is a structure $(A, +, \cdot, \times, \|\cdot\|)$ such that*

a. $(A, +, \cdot, \|\cdot\|)$ *is a Banach normed complex linear space;*

b. $(A, +, \cdot, \times)$ *is a algebra;*

c. it is satisfied the submultiplicity condition

$$\|x \times y\| \leq \|x\| \, \|y\|, \ \ \forall x, y \in A.$$

While the condition [c.] ensures that the internal multiplication is a continuous operation, the condition [b.] says that it is associative, that is, for all $x, y, z \in A$

$$x \times (y \times z) = (x \times y) \times z \ ,$$

and that are obeyed all the compatibility conditions:

. $(x+y)\times z = x\times z + y\times z$ and $x\times(y+z) = x\times y + x\times z$

. $\lambda \cdot (x \times y) = (\lambda \cdot x) \times y = x \times (\lambda \cdot y)$

If A contains an element e such that $e \times x = x \times e = x$, for every $x \in A$, and $\|e\| = 1$, we say that A is a Banach algebra with unit. When $x\times y = y\times x$, for every $x, y \in A$, we say that A is a commutative Banach algebra.

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Definition 2 *We say that $f : [a, b] \to X$ is a regulated function if for every $t \in [a, b]$ there exist both one-sided limits $f(t+)$ and $f(t-)$ with the convention $f(a-) = f(a)$ and $f(b+) = f(b)$.*

We denote by $G([a, b], X)$ the Banach space of all X-valued regulated functions on $[a, b]$, with the uniform convergence norm $\|f\|_\infty = \sup\{\|f(t)\|_{\mathcal{A}},\ t \in [a, b]\}$. We begin proving that multiplication of X induces an internal multiplication in $G([a, b], X)$.

Lemma 1 *Let f and g be two regulated functions on $[a, b]$ with values in a Banach algebra A. Then the pointwise multiplication $[f \times g](t) = f(t) \times_A g(t)$, $t \in [a, b]$ is a regulated function on $[a, b]$.*

Proof: Since every regulated function is a bounded function, if f and g are regulated functions on $[a, b]$, there are real numbers M and N such that $\|f(t)\| \le M$ and $\|g(t)\| \le N$, for all $t \in [a, b]$. Let L be the real number $L = \max\{M, N\}$. Moreover, for every $\epsilon > 0$ there are partitions $R : a = r_0 < r_1 < .. < r_m = b$ and $S : a = s_0 < s_1 < .. < s_n = b$ of $[a, b]$ such that for all $r_{j-1} < \xi < \eta < r_j$ ($j = 1, 2, .., m$) and for all $s_{k-1} < \xi < \eta < s_k$ ($k = 1, 2, .., n$), we have

$$\|f(\xi) - f(\eta)\| < \frac{\epsilon}{2\,L} \quad \text{and} \quad \|g(\xi) - g(\eta)\| < \frac{\epsilon}{2\,L}.$$

We consider now the partition $T = R \cup S$, $T : a = t_0 < t_1 < .. < t_l = b$, and $\xi,\ \eta \in]t_{i-1}, t_i[$, $i = 1, 2, .., l$. Then

$$\begin{aligned}\|[f \times g](\xi) - [f \times g](\eta)\| &= \|f(\xi)\, g(\xi) - f(\eta)\, g(\eta)\| \\ &= \|f(\xi)\, g(\xi) - f(\xi)\, g(\eta) + f(\xi)\, g(\eta) - f(\eta)\, g(\eta)\| \\ &= \|f(\xi)\, [g(\xi) - g(\eta)] + g(\eta)\, [f(\xi) - f(\eta)]\,\| \\ &\le \|f(\xi)\|\|g(\xi) - g(\eta)\| + \|g(\eta)\|\|f(\xi) - f(\eta)\| < \epsilon\end{aligned}$$

and so $f \times g$ is a regulated function. ∎

As a consequence we have that the structure of Banach algebra is transferred to the space of regulated functions.

Theorem 1 *Suppose that A is a real Banach algebra with multiplication $x \times y$. Then $G([a, b], A)$ with multiplication $[f \times g](t) = f(t) \times g(t)$ is a real Banach algebra too.*

Proof: It is known that $G([a, b], A)$ is a Banach space with the norm of uniform convergence (see [4]). Moreover its unit element is the function $e(t) = e_A, t \in [a, b]$, and if $f, g \in G([a, b], A)$ they are bounded functions, and so $F = \|f\|([a, b]) = \{\|f(t)\| \ :\ t \in [a, b]\}$ and $G = \|g\|([a, b]) = \{\|g(t)\| \ :\ t \in [a, b]\}$ are bounded subsets of positive numbers of $\mathbb{R}$. The set

$$F\ G = \|f\|([a, b])\ \|g\|([a, b]) = \{\|f(t)\ g(t)\|\ ,\ t \in [a, b]\}$$

is a bounded set and $\sup(F\ G) = \sup(F)\ \sup(G)$. So we have

$$\begin{aligned}\|f \times g\|_\infty &= \sup\{\|f(t)\ g(t)\|,\ t \in [a, b]\} \\ &= \sup\{\|f(t)\|,\ t \in [a, b]\}\ \sup\{\|g(t)\|,\ t \in [a, b]\} \\ &= \|f\|_\infty \|g\|_\infty\end{aligned}$$

and then $\|f \times g\|_\infty \le \|f\|_\infty \|g\|_\infty$ ∎

We note that if A is commutative Banach algebra, then $G([a, b], A)$ is also commutative.

Let A, B be two Banach algebras with multiplications $\times_A$ and $\times_B$ respectively. We consider the complex linear space

$$\mathcal{L}(A, B) = \{T : A \to B :\ T \text{ is a bounded linear operator}\},$$

with the usual norm

$$\|T\| = \sup\left\{\|T(x)\|_B\ :\ x \in A,\ \|x\|_A \le 1\right\}\ , \tag{1}$$

and, if $T, S \in \mathcal{L}(A, B)$, we define the multiplication as

$$[T \cdot S](x) = T(x) \times_B S(x) \tag{2}$$

We have that for evey $T, S, F \in \mathcal{L}(A, B)$, $\lambda \in \mathbb{C}$ and $x \in A$,

$$[T\cdot(S\cdot F)](x) = T(x)\times_B (S\cdot F)(x) = T(x)\times_B [S(x)\times_B F(x)]$$

$$= [T(x) \times_B S(x)] \times_B F(x)$$

$$= [T \cdot S)](x) \times_B F(x) = [(T \cdot S) \cdot F](x)\ ,$$

$$[(T+S)\cdot F](x) = (T+S)(x)\times_B F(x) = [T(x)+S(x)]\times_B F(x)$$

$$= T(x) \times_B F(x) + S(x) \times_B F(x)$$

$$= [T \cdot F](x) + [S \cdot F](x) = [T \cdot F + S \cdot F](x)\ ,$$

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$$[\lambda\ (T \cdot S)](x) = \lambda\ (T \cdot S)(x) = \lambda\ [T(x) \times_{B} S(x)]$$

$$= [\lambda\ T(x)] \times_{B} S(x) = [\lambda\ T](x) \times_{B} S(x)$$

$$= [(\lambda\ T) \cdot S](x)\ ,$$

$$[(\lambda\ T) \cdot S](x) = (\lambda\ T)(x) \times_{B} S(x) = \lambda\ T(x) \times_{B} S(x)$$

$$= T(x) \times_{B} \lambda\ S(x) = T(x) \times_{B} (\lambda\ S)(x)$$

$$= [T \cdot (\lambda\ S)](x)\ .$$

So we have that

- $T \cdot (S \cdot F) = (T \cdot S) \cdot F$:
- $(T+S) \cdot F = T \cdot F + S \cdot F$ and $T \cdot (S+F) = T \cdot S + T \cdot F$:
- $\lambda(T \cdot S) = (\lambda T) \cdot S = T \cdot (\lambda S)$:

Moreover, $\|T \cdot S\| \leq \|T\|\|S\|$, for all $T, S \in \mathcal{L}(A, B)$. Indeed, since B is a Banach algebra, $\|T(x) \times_{B} S(x)\|_{B} \leq \|T(x)\|_{B} \|S(x)\|_{B}$, for all $x \in A$, and so, if

$C = \{\|T(x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$

and

$D = \{\|S(x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$,

then $C\ D = \{\|T(x)\|_{B} \|S(x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$ and $\sup C\ D = \sup C\ \sup D$,

$$\|T \cdot S\| = \sup \{\|[T \cdot S](x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$$

$$= \sup \{\|T(x) \times_{B} S(x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$$

$$\leq \sup \{\|T(x)\|_{B} \|S(x)\|_{B} :\ x \in A,\ \|x\|_{A} \leq 1\}$$

$$= \sup C\ D = \sup C\ \sup D = \|T\|\|S\|$$

The unit element of $\mathcal{L}(A, B)$ is the operator $1_{\mathcal{L}(A)} : A \rightarrow B$ such that $1_{\mathcal{L}(A)}(x) = 1_{B}$, for every $x \in A$. Of course, $\|1_{\mathcal{L}(A)}\| = 1$.

We summaryze this in

Lemma 2 *If A, B are two Banach algebras, then $\mathcal{L}(A, B)$ with the usual norm and the multiplication (2) is a Banach algebra. If B is a commutative algebra, so $\mathcal{L}(A, B)$.*

Besides, when $B = A$ the composition of operators $T \circ S$ is the natural multiplication on $\mathcal{L}(A)$. In this case, $[T \circ S](x) = T(S(x))$,

$$\|T \circ S\| = \sup \{\|[T \circ S](x)\|\ :\ x \in A,\ \|x\| \leq 1\} \leq \|T\|\|S\|$$

and we recall that if $dim A > 1$, then $\mathcal{L}(A)$ is non-commutative algebra.

We present now the notion of integral (in sense of Dushnik) that we use to decribe a perfomance criterion on the Banach algebra of regulated function. This kind of Stieltjes integral, finest than the Riemann-Stieltjes integral, is a convenient choice because, when the integrand function belongs to $G([a, b], A)$ and the integrator function is of bounded semivariation, the integral there exists. The original definition of the Riemann integral has been modified in several different extensions. T. J. Stieltjes generalized the Riemann integral defining an integration of a continuous integrand with respect a bounded variation integrator, instead of the variable of integration. B. Dushnik in turn considered a integrand modification that consists in restricting integrand values only to the open segments of corresponding partitions of the interval $[a, b]$. This is a special case of the weighted refinement integral.

Let A, B be two Banach algebras with multiplications $\times_{A}$ and $\times_{B}$ respectively and suppose that $\alpha \in SV([a, b], \mathcal{L}(A, B))$, the Banach space of all bounded semivariation functions $\alpha : [a, b] \rightarrow \mathcal{L}(A, B)$, and $f \in G([a, b], A)$. Then there exists the Dushnik integral (see [4] for details)

$$F_{\alpha}(f) = \int_{a}^{b} \cdot\, d\ \alpha(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^{;}),$$

where $\xi_i^{;} \in]t_{i-1}, t_i[$. Here the limit is take over the set of all partitions of the interval $[a, b]$, denoted by $\mathcal{D}_{[a,b]}$.

Note that $F_{\alpha} : G([a, b], A) \rightarrow B$ is a linear map between the Banach algebras $G([a, b], A)$ and B. Moreover have sense write and to ask about

$$F_{\alpha}(f) \times_{B} F_{\alpha}(g) = \int_{a}^{b} \cdot\, d\ \alpha(t) \cdot f(t) \times_{B} \int_{a}^{b} \cdot\, d\ \alpha(t) \cdot g(t) \in B.$$

$$F_{\alpha}(f \times_{G} g) = \int_{a}^{b} \cdot\, d\ \alpha(t) \cdot [f \times_{G} g]\,(t)$$

Note that

$$F_\alpha(f) \times_{_B} F_\alpha(g) \neq F_\alpha(f \times_{_G} g)$$

we have that F_α is not a homomorphism of Banach algebras. Another special case is when $B = G([a,b], A)$ and $F_\alpha(f) \in G([a,b], A)$,

$$[F_\alpha(f)](s) = \int_a^s \cdot\, d\,\alpha(t) \cdot f(t) \in A\ , \quad s \in [a,b]\ .$$

The special case $G([a,b], Quat)$ will be considered in the next sections.

3 Quaternions

As an example, we consider the finite dimensional Banach algebra of quaternions $Quat$ with the usual norm (the absolut value) and the multiplication defined below. This structure was introduced by W. R. Hamilton in 1843, historically is the first example of a non-commutative algebra, and is used in representation of 3D rotations and in graphic computation. A quaterniom is an element of the form

$$p = r + r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$

where r, r_x, r_y, r_z are real numbers. The number r is called the real part of the quaternion p and $r_x\vec{i}+r_y\vec{j}+r_z\vec{k}$ is its imaginary part. Moreover $\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$, $\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$, $\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$ and $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$

We can define in $Quat$ three operations: the addition, multiplication by scalar and internal multiplication. For this, we consider the quaternions $p = r + r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$ and $q = s + s_x\vec{i} + s_y\vec{j} + s_z\vec{k}$, the scalar $\lambda \in \mathbb{R}$ and define

$$p + q = r + s + (r_x + s_x)\ \vec{i} + (r_y + s_y)\ \vec{j} + (r_z + s_z)\ \vec{k}$$

$$\lambda\ p = \lambda\ r + \lambda\ r_x\ \vec{i} + \lambda\ r_y\ \vec{j} + \lambda\ r_z\ \vec{k}$$

$$\begin{aligned} p \times q = r\ s &- (r_x\ s_x + r_y\ s_y + r_z\ s_z) \\ &+ (r\ s_x + r_x\ s + r_y\ s_z - r_z\ s_y)\ \vec{i} \\ &+ (r\ s_y - r_x\ s_z + r_y\ s + r_z\ s_x)\ \vec{j} \\ &+ (r\ s_z + r_x\ s_y - r_y\ s_x + r_z\ s)\ \vec{k}. \end{aligned}$$

Then $e = (1,0,0,0)$ is the unit element of $Quat$, the inverse of the quaternion p is the quaternion p^{-1} such that $p \times p^{-1} = e = p^{-1} \times p$, its conjugated is the quaternion $\bar{p} = r - r_x\vec{i} - r_y\vec{j} - r_z\vec{k}$, and the norm of p is defined as

$$\|p\| = \sqrt{r^2 + r_y^2 + r_y^2 + r_z^2}.$$

When $\|p\| = 1$, we say that p is an unitary quaternion, and in this case, it is easy to see that $\bar{p} = p^{-1}$. The vectorial notation $p = (r, \vec{u})$, where $r \in \mathbb{R}$ and $\vec{u} = (r_x, r_y, r_z) \in V^3$ (here we use that $Quat = \mathbb{R} \oplus V^3$), simplifies significantly the notation. Of course, $p + q = (r, \vec{u}) + (s, \vec{v}) = (r+s, \vec{u}+\vec{v})$ and $\lambda\, p = \lambda\,(r, \vec{u}) = (\lambda\, r, \lambda\, \vec{u})$. If $\vec{u} \cdot \vec{v}$ and $\vec{u} \wedge \vec{v}$ are respectively the scalar and vectorial product in V^3, then the multiplication can be expressed as

$$p \times q = (r, \vec{u}) \times (s, \vec{v}) = (r\ s - \vec{u} \cdot \vec{v}, r\ \vec{v} + s\ \vec{u} + \vec{u} \wedge \vec{v})$$

Moreover, $\|p\| = \sqrt{r^2 + \|\vec{u}\|^2}$ and $\bar{p} = (r, -\vec{u})$.

We now describe how the quaternions can be used to represent rotations. First suppose that $q = (s, \vec{v})$ is an unitary quaternion, i. e., $\|q\| = \sqrt{s^2 + \|\vec{v}\|^2} = 1$ and $q\ \bar{q} = 1$. Then there exists a angle θ such that $s = \cos\theta$ and $\|\vec{v}\| = \theta$. So, if $\|\vec{n}\| = 1$, we can to write

$$q = (s, \vec{v}) = (\cos\theta, \sin\theta\ \vec{n}). \tag{3}$$

Let $P = (r_x, r_y, r_z)$ be a point. If we denote by $\vec{r} = (r_x, r_y, r_z)$, we can associate to P the quaternion $p = (0, \vec{r})$. A rotation in the anti-clockwise direction at this point around the axis determined by the unitary vector $\vec{n}$ can be represented as

$$R_q(p) = q\ p\ q^{-1} \tag{4}$$

Since $q^{-1} = \bar{q}$, we have that

$$R_q(p) = \big(\ 0, s^2\vec{r} - (\vec{v} \cdot \vec{v})\vec{r} + 2(\vec{v} \cdot \vec{r})\vec{v} + 2s\vec{v} \wedge \vec{r}\ \big) \tag{5}$$

We use now (3) to conclude that

$$\begin{aligned} R_q(p) = \big(\ 0, s^2\ \vec{r} - (\cos 2\theta)\ \vec{r} + (1 - \cos 2\theta)\ (\vec{n}\ \cdot\ \vec{r})\vec{n} \\ + \sin 2\theta\ \vec{n} \wedge \vec{r}\ \big) \end{aligned} \tag{6}$$

As an example we take a body in the position $r = (r_x, r_y, r_z) = (1,2,1)$, and so $p = (0,1,2,1)$, and apply a rotation of angle $\theta = \pi/3$ around the axis determined by $\vec{n} = 1/2(1,1,\sqrt{2})$. Then $s = \cos\pi/6 = \sqrt{3}/2$,

$$\begin{aligned} q &= (s, \vec{v}) = (\cos\pi/6, \sin\pi/6\ \vec{n}) \\ &= (\sqrt{3}/2, (1/4)(1,1,\sqrt{2})) = (\sqrt{3}/2, (1/4, 1/4, \sqrt{2}/4)). \end{aligned}$$

We found

$$(\vec{n} \cdot \vec{r})\vec{n} = ([3 + \sqrt{2}]/4)(1,1,\sqrt{2})$$

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and

$\vec{n} \wedge \vec{r} = (1/2)(1-\sqrt{2}, \sqrt{2}-1, 1/2)$,

and, by (6)

$$R_q(p) = \left(0, 1 - \frac{(3-\sqrt{2})\sqrt{3}}{8},\right.$$

$$\left.\frac{1}{2} + \frac{3\sqrt{2}}{4} - \frac{(3-\sqrt{2})\sqrt{3}}{8}, \frac{(2-\sqrt{3})(3\sqrt{2}+2)}{8}\right)$$

4 Functions $Quat$-valued and Integrals

Here we start the application of the notions and results of the previous sections in the special case when $I = [0, T]$ and $\mathcal{A} = Quat$. Let $p : [0, T] \to Quat$ be a function. Then, for $t \in [0, T]$,

$$p(t) = r(t) + r_x(t)\vec{i} + r_y(t)\vec{j} + r_z(t)\vec{k},$$

where r, r_x, r_y, r_z are real functions on $[0, T]$. We think these kind of functions as a strategies, over time, of rigid motions (translations and rotations) of bodies. We recall first that $dim(Quat) = 4$ and so, if $p \in G([0, T], Quat)$ then p is differentiable a.e. . In vectorial notation we have

$$p(t) = (r(t), \vec{u}(t)),$$

$\vec{u}(t) = (r_x(t), r_y(t), r_z(t)) \in V^3$. Suppose that $p, q : [0, T] \to Quat$ are two regulated functions. Then by Lemma 1 we have that $p \times q$ is a regulated function, that is, $p \times q \in G([0, T], Quat)$. In vectorial form we have $p(t) = (r(t), \vec{u}(t))$ and $q(t) = (s(t), \vec{v}(t))$, where $\vec{u}(t) = (r_x(t), r_y(t), r_z(t)) \in V^3$, $\vec{v}(t) = (s_x(t), s_y(t), s_z(t)) \in V^3$ and $r, s : [0, T] \to \mathbb{R}$,

$[p \times q](t) = (r(t), \vec{u}(t)) \times (s(t), \vec{v}(t))$

$= (r(t)\ s(t) - \vec{u}(t) \cdot \vec{v}(t), r(t)\ \vec{v}(t) + s(t)\ \vec{u}(t) + \vec{u}(t) \wedge \vec{v}(t))$

We observe that

$$\|p\|_\infty = \sup\{\|p(t)\| \ : \ t \in [a, b]\}$$

$$= \sup\{\| \sqrt{r^2(t) + \|\vec{u}(t)\|^2} \ : \ t \in [0, T]\},$$

denote the multiplicative inverse of $p \in G([0, T], Quat)$ by $1/p$,

$$(\frac{1}{p})(t) = \frac{1}{p(t)} \ , p(t) \neq 0 \ , \ t \in [0, T] \ ,$$

and define

$$\bar{p}(t) = (r(t), -\vec{u}(t))$$

Example 1 If $p(t) = (0, \vec{u}(t))$ for example, we back to Section 3 and think $p(t)$ as a position of a body in space, at each time t. When $\|\vec{u}\| = 1$ then $p(t)$ describe points belong to the unitary sphere and $\bar{p}(t)$ is the antipode point of $p(t)$.

Example 2 We fix $c, d \in]0, T[$ and $q \in Quat$. Define $p : [0, T] \to Quat$, as

$$p(t) = \begin{cases} q, \ , \ t \in [c, d[\ , \ , \\ 0 \ , \ otherwise. \end{cases}$$

Then

$$\|p\|_\infty = \|q\|.$$

Example 3 We fix $q \in Quat$ such that $\|q\| = 1$. Consider a function $p \in G([0, T], Quat)$ and define $R_q : [0, T] \to Quat$ as

$$[R_q p](t) = [q\ p\ \bar{q}](t).$$

Suppose that $p \in G([0, T], Quat)$ satisfy some restriction, a differential equation a. e. or an integral equation, for example. We can define a performance criterion using a functional defined on the set of such regulated functions. Formally we take $F : p \in G([0, T], Quat) \to F(p) \in \mathbb{R}$. For example, $F(p) = p(T)$ or $F(p) = \|p\|_Q$. A special case for us is when F is a linear functional. Then we a representation of F in an integral (in sense of Dushnik of Section 2) form.

$$F_\beta(p) = \int_0^T \cdot\ d\ \beta(\eta) \cdot p(\eta)$$

Moreover, in examples above we have

$$|F_\beta(p)| \leq BV[\beta]\ \|p\|_\infty \ ,$$

where $BV[\beta]$ is the bounded variation of β. Recalling that $Quat = \mathbb{R} \oplus V^3 \approx \mathbb{R}^4$, we consider a regulated function $p : [0, T] \to \mathbb{R} \oplus V^3$ and a bounded variation function $\beta : [0, T] \to \mathcal{L}(\mathbb{R} \oplus \mathbb{V}^3, \mathbb{R})$. If we use matricial notation,

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$$\beta(t) = \begin{pmatrix} \gamma(t) & \gamma_x(t) & \gamma_y(t) & \gamma_z(t) \end{pmatrix}, \; p(t) = \begin{pmatrix} r(t) \\ r_x(t) \\ r_y(t) \\ r_z(t) \end{pmatrix}$$

and

$$\beta(t)\cdot p(t) = \gamma(t)\, r(t) + \gamma_x(t)\, r_x(t) + \gamma_y(t)\, r_y(t) + \gamma_z(t)\, r_z(t)$$

We suppose in the next that there exists the integrals. Then (cf. [8], section 1.2)

$$\int_0^T \cdot\, d\,\beta(\eta)\cdot p(\eta) = \int_0^T \cdot\, d\,\gamma(\eta)\; r(\eta) + \int_0^T \cdot\, d\,\gamma_x(\eta)\; r_x(\eta)$$

$$+ \int_0^T \cdot\, d\,\gamma_y(\eta)\; r_y(\eta) + \int_0^T \cdot\, d\,\gamma_z(\eta)\; r_z(\eta)$$

For example, recalling that the unit element of $G([0,T], Quat)$ is the function $e(t) = e_{_Q}$, where $e_{_Q} = (1,0,0,0)$. then

$$\int_0^T \cdot\, d\,\beta(\eta)\cdot e(\eta) = \int_0^T \cdot\, d\,\gamma(\eta)\; 1 = \gamma(T) - \gamma(0)$$

We take for illustration the integrator function $\beta : [0,T] \to \mathcal{L}(Quat, \mathbb{R})$ given by

$$\beta(t) = \begin{cases} F, \;,\; t \in [0, T[\;,\;, \\ 0\;,\; t = T. \end{cases}$$

where F is a bounded linear functional on $Quat$. Then

$$\int_0^T \cdot\, d\,\beta(\eta)\cdot p(\eta) = p(T-) - p(0+).$$

Here

$$p(T-) = \lim_{t\uparrow T} p(t) \quad \text{and} \quad p(0+) = \lim_{t\downarrow 0} p(t).$$

5 Conclusions and Future Work

This paper is part of an effort to find examples of regulated functions with values on Banach algebras. In the future we hope will be the conection of this subject with 3D-rotations.

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