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Optimal Inventory Control under Parametric Uncertainty via Cumulative Customer Demand

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***Abstract*— Most models, which are used for solving inventory control problems, are developed in the literature under the assumptions that the parameter values of the models are known with certainty. When these models are applied to solve real-world problems, the parameters are estimated and then treated as if they were the true values. The risk associated with using estimates rather than the true parameters is called estimation risk and is often ignored. In this paper, we consider stochastic inventory control problems which are invariant with respect to a certain group of transformations. If a given decision problem admits a sufficient statistic, it is well known that the class of invariant rules based on the sufficient statistic is essentially complete (under some assumptions) in the class of all invariant rules. If, in this case, there exists an optimal invariant rule among invariant rules based on sufficient statistic, it is optimal among all invariant rules. The primary purpose of this paper is to introduce the idea of cumulative customer demand in inventory control problems to deal with the order statistics from the underlying distribution. Transformations of the performance index based on pivotal quantities and ancillary statistics allow one to eliminate unknown parameters from the problem and to find the optimal statistical decisions for stochastic inventory control under parametric uncertainty. Illustrative examples are given.**

***Index Terms* — Stochastic inventory control, uncertainty, optimization**

I. Introduction

A large number of problems in production planning and scheduling, location, transportation, finance, and engineering design require that decisions be made in the presence of uncertainty. Most of the inventory management literature assumes that demand distributions are specified explicitly. However, in many practical situations, the true demand distributions are not known, and the only information available may be a time-series of historic demand data. When the demand distribution is unknown, one may either use a parametric approach (where it is assumed that the demand distribution belongs to a parametric family of distributions) or a non-parametric approach (where no assumption regarding the parametric form of the unknown demand distribution is made).

Under the parametric approach, one may choose to estimate the unknown parameters or choose a prior distribution for the unknown parameters and apply the Bayesian approach to incorporating the demand data available. Scarf [1] and Karlin [2] consider a Bayesian framework for the unknown demand distribution. Specifically, assuming that the demand distribution belongs to the family of exponential distributions, the demand process is characterized by the prior distribution on the unknown parameter. Further extension of this approach is presented in [3]. Application of the Bayesian approach to the censored demand case is given in [4-5]. Parameter estimation is first considered in [6] and recent developments are reported in [7-8]. Liyanage and Shanthikumar [9] propose the concept of operational statistics and apply it to a single period newsvendor inventory control problem.

Within the non-parametric approach, either the empirical distribution or the bootstrapping method (e.g. see [10]) can be applied with the available demand data to obtain an inventory control policy.

In this paper we consider the case, where it is known that the demand distribution function belongs to a parametric family of distribution functions. However, unlike in the Bayesian approach, we do not assume any prior knowledge on the parameter values.

Conceptually, it is useful to distinguish between "new-sample" inventory control, "within-sample" inventory control, and "new-within-sample" inventory control.

For the new-sample inventory control process, the data from a past sample of customer demand are used to make a statistical decision on a future time period for the same inventory control process.

For the within-sample inventory control process, the problem is to make a statistical decision on a future time period for the same inventory control process based on early data from that sample of customer demand.

For the new-within-sample inventory control process, the problem is to make a statistical decision on a future time period for the inventory control process based on early data from that sample of customer demand as well as on a past data sample of customer demand from the same process.

In this paper, we obtain optimal statistical decisions under parametric uncertainty for the within-sample inventory control process.

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II. Within – Sample Inventory Control Process

A. Mathematical Preliminaries

Theorem 1. Let $X_1 \le ... \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from a continuous distribution with some probability density function $f_\theta(x)$ and distribution function $F_\theta(x)$, where θ is a parameter (in general, vector). Then the joint probability density function of $X_1 \le ... \le X_k$ and the lth order statistics X_l $(1 \le k < l \le m)$ is given by

$$f_\theta(x_1,...,x_k,x_l) = f_\theta(x_1,...,x_k) f_\theta(x_l \mid x_k), \qquad (1)$$

where

$$f_\theta(x_1,...,x_k) = \frac{m!}{(m-k)!}\prod_{i=1}^{k} f_\theta(x_i)[1-F_\theta(x_k)]^{m-k}, \qquad (2)$$

$$f_\theta(x_l \mid x_k) = \frac{(m-k)!}{(l-k-1)!(m-l)!}\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k-1}$$

$$\times\left[1-\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{m-l}\frac{f_\theta(x_l)}{1-F_\theta(x_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j$$

$$\times\left[\frac{1-F_\theta(x_l)}{1-F_\theta(x_k)}\right]^{m-l+j}\frac{f_\theta(x_l)}{1-F_\theta(x_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{m-l}\binom{m-l}{j}(-1)^j$$

$$\times\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k-1+j}\frac{f_\theta(x_l)}{1-F_\theta(x_k)} \qquad (3)$$

represents the conditional probability density function of X_l given X_k=x_k.

Proof. The joint density of $X_1 \le ... \le X_k$ and X_l is given by

$$f_\theta(x_1,...,x_k,x_l) = \frac{m!}{(l-k-1)!(m-l)!}\prod_{i=1}^{k} f_\theta(x_i)$$

$$\times[F_\theta(x_l)-F_\theta(x_k)]^{l-k-1} f_\theta(x_l)[1-F_\theta(x_l)]^{m-l}$$

$$= f_\theta(x_1,...,x_k) f_\theta(x_l \mid x_k). \qquad (4)$$

It follows from (4) that

$$f_\theta(x_l \mid x_1,...,x_k) = \frac{f_\theta(x_1,...,x_k,x_l)}{f_\theta(x_1,...,x_k)} = f_\theta(x_l \mid x_k), \qquad (5)$$

i.e., the conditional distribution of X_l, given $X_i = x_i$ for all i = 1,…, k, is the same as the conditional distribution of X_l, given only $X_k = x_k$, which is given by (5). This ends the proof. □

Corollary 1.1. The conditional probability distribution function of X_l given X_k=x_k is

$$P_\theta\{X_l \le x_l \mid X_k = x_k\} = 1-\frac{(m-k)!}{(l-k-1)!(m-l)!}$$

$$\times\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\frac{1-F_\theta(x_l)}{1-F_\theta(x_k)}\right]^{m-l+1+j}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{m-l}\binom{m-l}{j}\frac{(-1)^j}{l-k+j}\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k+j}. \qquad (6)$$

B. Exponential Distribution

In order to use the results of Theorem 1, we consider, for illustration, the exponential distribution with the probability density function

$$f_\theta(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \quad x \ge 0, \quad \theta > 0, \qquad (7)$$

and the probability distribution function

$$F_\theta(x) = 1-\exp\left(-\frac{x}{\theta}\right), \quad x \ge 0, \quad \theta > 0. \qquad (8)$$

Theorem 2. Let $X_1 \le ... \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution (7). Then the conditional probability density function of the lth order statistics X_l $(1 \le k < l \le m)$ given $X_k = x_k$ is

$$g_\theta(x_l \mid x_k) = \frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j$$

$$\times\frac{1}{\theta}\exp\left(-\frac{(m-l+1+j)(x_l-x_k)}{\theta}\right)$$

$$= \frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{m-l}\binom{m-l}{j}(-1)^j$$

$$\times\frac{1}{\theta}\left[1-\exp\left(-\frac{x_l-x_k}{\theta}\right)\right]^{l-k-1+j}\exp\left(\frac{x_l-x_k}{\theta}\right), \qquad (9)$$

and the conditional probability distribution function of the lth order statistics X_l given $X_k = x_k$ is

$$P_\theta\{X_l \le x_l \mid X_k = x_k\}$$

$$= 1-\frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}$$

$$\times\frac{(-1)^j}{m-l+1+j}\exp\left(-\frac{(m-l+1+j)(x_l-x_k)}{\theta}\right)$$

$$= \frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{m-l}\binom{m-l}{j}\frac{(-1)^j}{l-k+j}$$

$$\times\left[1-\exp\left(-\frac{x_l-x_k}{\theta}\right)\right]^{l-k+j}. \qquad (10)$$

Proof. It follows from (3) and (6), respectively. □

Theorem 3. Let $X_1 \le ... \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution (11), where the parameter θ is unknown. Then the predictive probability density function of the lth order statistics X_l $(1 \le k < l \le m)$ is given by

$$g_{s_k}(x_l \mid x_k) = \frac{k}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j$$

$$\times\left[1+(m-l+1+j)\frac{x_l-x_k}{s_k}\right]^{-(k+1)}\frac{1}{s_k},\quad x_l\geq x_k, \qquad (11)$$

where

$$S_k=\sum_{i=1}^{k}X_i+(m-k)X_k \qquad (12)$$

is the sufficient statistic for θ, and the predictive probability distribution function of the lth order statistics X_l is given by

$$P_{s_k}\{X_l\leq x_l\mid X_k=x_k\}$$

$$=1-\frac{1}{\mathrm{B}(l-k,(m-l+1))}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$

$$\times\left[1+(m-l+1+j)\frac{x_l-x_k}{s_k}\right]^{-k}. \qquad (13)$$

Proof. Using the technique of invariant embedding [11-13], we reduce (9) to

$$g_\theta(x_l\mid x_k)=\frac{1}{\mathrm{B}(l-k,(m-l+1))}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j$$

$$\times v\exp\left(-\frac{(m-l+1+j)(x_l-x_k)}{s_k}v\right)\frac{1}{s_k}=g_{s_k}(x_l\mid x_k,v), \qquad (14)$$

where

$$V=\frac{S_k}{\theta} \qquad (15)$$

is the pivotal quantity, the probability density function of which is given by

$$f(v)=\frac{1}{\Gamma(k)}v^{k-1}\exp(-v),\quad v\geq 0. \qquad (16)$$

Then

$$g_{s_k}(x_l\mid x_k)=E\{g_{s_k}(x_l\mid x_k,v)\}$$

$$=\int_0^\infty g_{s_k}(x_l\mid x_k,v)f(v)dv=g_{s_k}(x_l\mid x_k). \qquad (17)$$

This ends the proof. □

Corollary 3.1. If $l=k+1$,

$$g_{s_k}(x_{k+1}\mid x_k)=k(m-k)\left[1+(m-k)\frac{x_{k+1}-x_k}{s_k}\right]^{-(k+1)}\frac{1}{s_k},$$

$$x_{k+1}\geq x_k,\quad 1\leq k\leq m-1, \qquad (18)$$

and

$$P_{s_k}\{X_{k+1}\leq x_{k+1}\mid X_k=x_k\}=1-\left[1+(m-k)\frac{x_{k+1}-x_k}{s_k}\right]^{-k}, \qquad (19)$$

C. Cumulative Customer Demand

The primary purpose of this paper is to introduce the idea of cumulative customer demand in inventory control problems to deal with the order statistics from the underlying distribution. It allows one to use the above results to improve statistical decisions for inventory control problems under parametric uncertainty.

Assumptions. The customer demand at the ith period represents a random variable Y_i, $i\in\{1, \ldots, m\}$. It is assumed (for the cumulative customer demand) that the random variables

$$X_1=Y_1,\ldots,X_k=\sum_{i=}^{k}Y_i,\ldots,X_l=\sum_{i=1}^{l}Y_i,\ldots,X_m=\sum_{i=1}^{m}Y_i \qquad (20)$$

represent the order statistics ($X_1\leq\ldots\leq X_m$) from the exponential distribution (7).

Inferences. For the above case, we have the following inferences.

Conditional probability density function *of* Y_{k+1}, $k\in\{1, \ldots, m-1\}$, is given by

$$g_\theta(y_{k+1}\mid k)=\frac{m-k}{\theta}\exp\left(-\frac{(m-k)y_{k+1}}{\theta}\right),\quad y_{k+1}\geq 0; \qquad (21)$$

Conditional probability distribution function of Y_{k+1}, $k\in\{1, \ldots, m-1\}$, is given by

$$G_\theta\{y_{k+1}\mid k\}=1-\exp\left(-\frac{(m-k)y_{k+1}}{\theta}\right). \qquad (22)$$

Conditional probability density function of $Z_m=\sum_{i=k+1}^{m}Y_i$ is given by

$$g_\theta(z_m\mid k)=(m-k)\frac{1}{\theta}\left[1-\exp\left(-\frac{z_m}{\theta}\right)\right]^{m-k-1}\exp\left(-\frac{z_m}{\theta}\right),$$

$$z_m\geq 0,\quad 1\leq k\leq m-1; \qquad (23)$$

Conditional probability distribution function of $Z_m=\sum_{i=k+1}^{m}Y_i$ is given by

$$G_\theta(z_m\mid k)=\left[1-\exp\left(-\frac{z_m}{\theta}\right)\right]^{m-k},\quad 1\leq k\leq m-1. \qquad (24)$$

III. Stochastic Inventory Control and its Optimization

This section deals with inventory items that are in stock during a single time period. At the end of the period, leftover units, if any, are disposed of, as in fashion items. Two models are considered. The difference between the two models is whether or not a setup cost is incurred for placing an order. The symbols used in the development of the models include:

c = setup cost per order,
c_1= holding cost per held unit during the period,
c_2= penalty cost per shortage unit during the period,
$g_\theta(y_{k+1}|k)$= conditional probability density function of customer demand, Y_{k+1}, during the (k+1)th period,
θ= parameter (in general, vector),
u = order quantity,
q = inventory on hand before an order is placed.

A. No-Setup Model (Newsvendor Model)

This model is known in the literature as the *newsvendor* model (the original classical name is the *newsboy* model). It deals with stocking and selling newspapers and periodicals. The assumptions of the model are:

1. Demand occurs instantaneously at the start of the period immediately after the order is received.

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2. No setup cost is incurred.

The model determines the optimal value of u that minimizes the sum of the expected holding and shortage costs. Given optimal u (= u^*), the inventory policy calls for ordering $u^* - q$ if $q < u^*$; otherwise, no order is placed.

If $Y_{k+1} \le u$, the quantity $u - Y_{k+1}$ is held during the (k+1)th period. Otherwise, a shortage amount $Y_{k+1} - u$ will result if $Y_{k+1} > u$. Thus, the cost per the (k+1)th period is

$$C(u) = \begin{cases} c_1 \dfrac{u - Y_{k+1}}{\theta} & \text{if} \quad Y_{k+1} \le u, \\ c_2 \dfrac{Y_{k+1} - u}{\theta} & \text{if} \quad Y_{k+1} > u. \end{cases} \tag{25}$$

The expected cost for the (k+1)th period, $E_\theta\{C(u)\}$, is expressed as

$$E_\theta\{C(u)\} = \frac{1}{\theta}\left(c_1 \int_0^u (u - y_{k+1}) g_\theta(y_{k+1} \mid k) dy_{k+1} + c_2 \int_u^\infty (y_{k+1} - u) g_\theta(y_{k+1} \mid k) dy_{k+1} \right). \tag{26}$$

The function $E_\theta\{C(u)\}$ can be shown to be convex in u, thus having a unique minimum. Taking the first derivative of $E_\theta\{C(u)\}$ with respect to u and equating it to zero, we get

$$\frac{1}{\theta}\left(c_1 \int_0^u g_\theta(y_{k+1} \mid k) dy_{k+1} - c_2 \int_u^\infty g_\theta(y_{k+1} \mid k) dy_{k+1} \right) = 0 \tag{27}$$

or

$$c_1 P_\theta\{Y_{k+1} \le u\} - c_2(1 - P_\theta\{Y_{k+1} \le u\}) = 0 \tag{28}$$

or

$$P_\theta\{Y_{k+1} \le u\} = \frac{c_2}{c_1 + c_2}. \tag{29}$$

It follows from (21), (22), (26), and (29) that

$$u^* = \frac{\theta}{m-k} \ln\left(1 + \frac{c_2}{c_1}\right) \tag{30}$$

and

$$E_\theta\{C(u^*)\} = \frac{1}{\theta}\left(c_2 E_\theta\{Y_{k+1}\} - (c_1 + c_2) \int_0^{u^*} y_{k+1} g_\theta(y_{k+1} \mid k) dy_{k+1} \right) = \frac{c_1}{m-k} \ln\left(1 + \frac{c_2}{c_1}\right). \tag{31}$$

B. Parametric Uncertainty

Consider the case when the parameter θ is unknown. To find the best invariant decision rule u^{BI}, we use the invariant embedding technique [11-13] to transform (25) to the form, which is depended only on the pivotal quantities V, V_1, and the ancillary factor η. In statistics, a pivotal quantity or pivot is a function of observations and unobservable parameters whose probability distribution does not depend on unknown parameters. Note that a pivotal quantity need not be a statistic—the function and its value can depend on parameters of the model, but its distribution must not. If it is a statistic, then it is known as an ancillary statistic.

Transformation of $C(u)$ based on the pivotal quantities V, V_1 is given by

$$C^{(1)}(\eta) = \begin{cases} c_1(\eta V - V_1) & \text{if} \quad V_1 \le \eta V, \\ c_2(V_1 - \eta V) & \text{if} \quad V_1 > \eta V, \end{cases} \tag{32}$$

where

$$\eta = \frac{u}{S_k}, \tag{33}$$

$$V_1 = \frac{Y_{k+1}}{\theta} \sim g(v_1 \mid k) = (m-k)\exp[-(m-k)v_1], \quad v_1 \ge 0. \tag{34}$$

Then $E\{C^{(1)}(\eta)\}$ is expressed as

$$E\{C^{(1)}(\eta)\} = \int_0^\infty \left(c_1 \int_0^{\eta v} (\eta v - v_1) g(v_1 \mid k) dv_1 + c_2 \int_{\eta v}^\infty (v_1 - \eta v) g(v_1 \mid k) dv_1 \right) f(v) dv. \tag{35}$$

The function $E\{C^{(1)}(\eta)\}$ can be shown to be convex in η, thus having a unique minimum. Taking the first derivative of $E\{C^{(1)}(\eta)\}$ with respect to η and equating it to zero, we get

$$\int_0^\infty v \left(c_1 \int_0^{\eta v} g(v_1 \mid k) dv_1 - c_2 \int_{\eta v}^\infty g(v_1 \mid k) dv_1 \right) f(v) dv = 0 \tag{36}$$

or

$$\int_0^\infty v P(V_1 \le \eta v) f(v) dv \Big/ \int_0^\infty v f(v) dv = \frac{c_2}{c_1 + c_2}. \tag{37}$$

It follows from (33), (35), and (37) that the optimum value of η is given by

$$\eta^* = \frac{1}{m-k}\left[\left(1 + \frac{c_2}{c_1}\right)^{1/(k+1)} - 1 \right], \tag{38}$$

the best invariant decision rule is

$$u^{\mathrm{BI}} = \eta^* S_k = \frac{S_k}{m-k}\left[\left(1 + \frac{c_2}{c_1}\right)^{1/(k+1)} - 1 \right], \tag{39}$$

and the expected cost, if we use u^{BI}, is given by

$$E_\theta\{C(u^{\mathrm{BI}})\} = \frac{c_1(k+1)}{m-k}\left[\left(1 + \frac{c_2}{c_1}\right)^{1/(k+1)} - 1 \right] = E\{C^{(1)}(\eta^*)\}. \tag{40}$$

It will be noted that, on the other hand, the invariant embedding technique [11-13] allows one to transform equation (26) as follows:

$$E_\theta\{C(u)\} = \frac{1}{\theta}\left(c_1 \int_0^u (u - y_{k+1}) g_\theta(y_{k+1} \mid k) dy_{k+1} + c_2 \int_u^\infty (y_{k+1} - u) g_\theta(y_{k+1} \mid k) dy_{k+1} \right)$$

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$$=\frac{1}{s_k}\left(c_1\int_0^u (u-y_{k+1})v^2(m-k)\exp\left(-\frac{v(m-k)y_{k+1}}{s_k}\right)\frac{1}{s_k}dy_{k+1}\right.$$
$$\left.+c_2\int_u^\infty (y_{k+1}-u)v^2(m-k)\exp\left(-\frac{v(m-k)y_{k+1}}{s_k}\right)\frac{1}{s_k}dy_{k+1}\right). \quad (41)$$

Then it follows from (41) that

$$E\{E_\theta\{(C(u)\}\}=\int_0^\infty E_\theta\{C(u)\}f(v)dv=E_{s_k}\{C^{(1)}(u)\}, \quad (42)$$

where

$$E_{s_k}\{C^{(1)}(u)\}=\frac{k}{s_k}\left(c_1\int_0^u (u-y_{k+1})g^\bullet_{s_k}(y_{k+1}\,|\,k)dy_{k+1}\right.$$
$$\left.+c_2\int_u^\infty (y_{k+1}-u)g^\bullet_{s_k}(y_{k+1}\,|\,k)dy_{k+1}\right) \quad (43)$$

represents the expected prediction cost for the (k+1)th period. It follows from (57) that the cost per the (k+1)th period is reduced to

$$C^{(2)}(u)=\begin{cases} c_1\dfrac{u-Y_{k+1}}{s_k/k} & \text{if} \quad Y_{k+1}\le u, \\ c_2\dfrac{Y_{k+1}-u}{s_k/k} & \text{if} \quad Y_{k+1}>u, \end{cases} \quad (44)$$

and the predictive probability density function of Y_{k+1} (compatible with (26)) is given by

$$g^\bullet_{s_k}(y_{k+1}\,|\,k)=(k+1)(m-k)\left[1+(m-k)\frac{y_{k+1}}{s_k}\right]^{-(k+2)}\frac{1}{s_k},$$
$$y_{k+1}\ge 0. \quad (45)$$

Minimizing the expected prediction cost for the (k+1)th period,

$$E_{s_k}\{C^{(2)}(u)\}=\frac{k}{s_k}\left(c_1\int_0^u (u-y_{k+1})g^\bullet_{s_k}(y_{k+1}\,|\,k)dy_{k+1}\right.$$
$$\left.+c_2\int_u^\infty (y_{k+1}-u)g^\bullet_{s_k}(y_{k+1}\,|\,k)dy_{k+1}\right), \quad (46)$$

with respect to u, we obtain u^{BI} immediately, and

$$E_{s_k}\{C^{(2)}(u^{\mathrm{BI}})\}=\frac{c_1(k+1)}{m-k}\left[\left(1+\frac{c_2}{c_1}\right)^{1/(k+1)}-1\right]. \quad (47)$$

It should be remarked that the cost per the (k+1)th period, $C^{(2)}(u)$, can also be transformed to

$$C^{(3)}(\eta)=\begin{cases} c_1k\left(\dfrac{u}{s_k}-\dfrac{Y_{k+1}}{s_k}\right) & \text{if} \quad \dfrac{Y_{k+1}}{s_k}\le\dfrac{u}{s_k} \\ c_2k\left(\dfrac{Y_{k+1}}{s_k}-\dfrac{u}{s_k}\right) & \text{if} \quad \dfrac{Y_{k+1}}{s_k}>\dfrac{u}{s_k} \end{cases}$$

$$=\begin{cases} c_1k(\eta-W) & \text{if} \quad W\le\eta \\ c_2k(W-\eta) & \text{if} \quad W>\eta, \end{cases} \quad (48)$$

where the probability density function of the ancillary statistic W (compatible with (40)) is given by

$$g^\circ_{s_k}(w\,|\,k)=(k+1)(m-k)[1+(m-k)w]^{-(k+2)}, \quad w\ge 0. \quad (49)$$

Then the best invariant decision rule $u^{\mathrm{BI}}=\eta^* S_k$, where η^* minimizes

$$E\{C^{(3)}(\eta)\}=k\left(c_1\int_0^\eta (\eta-w)g^\circ(w\,|\,k)dw\right.$$
$$\left.+c_2\int_\eta^\infty (w-\eta)g^\circ(w\,|\,k)dw\right). \quad (50)$$

C. Comparison of Statistical Decision Rules

For comparison, consider the maximum likelihood decision rule that may be obtained from (30),

$$u^{\mathrm{ML}}=\frac{\hat\theta}{m-k}\ln\left(1+\frac{c_2}{c_1}\right)=\eta_j^{\mathrm{ML}}S_k, \quad (51)$$

where $\hat\theta=S_k/k$ is the maximum likelihood estimator of θ,

$$\eta^{\mathrm{ML}}=\frac{1}{m-k}\ln\left(1+\frac{c_2}{c_1}\right)^{1/k}. \quad (52)$$

Since u^{BI} and u^{ML} belong to the same class,

$$\mathscr{C}=\{u: u=\eta S_k\}, \quad (53)$$

it follows from the above that u^{ML} is inadmissible in relation to u^{BI}. If, say, k=1 and c_2/c_1=100, we have that

$$\mathrm{Rel.eff.}\{u^{\mathrm{ML}}, u^{\mathrm{BI}}, \theta\}$$
$$=E_\theta\{C(u^{\mathrm{BI}})\}\big/E_\theta\{C(u^{\mathrm{ML}})\}=0.838. \quad (54)$$

Thus, in this case, the use of u^{BI} leads to a reduction in the expected cost of about 16.2 % as compared with u^{ML}. The absolute expected cost will be proportional to θ and may be considerable.

D. Setup Model (s–S Policy)

The present model differs from the one in Section C in that a setup cost c is incurred. Using the same notation, the total expected cost per the (k+1)th period is

$$E_\theta\{\overline{C}(u)\}=c+E_\theta\{C(u)\}$$
$$=c+\frac{1}{\theta}\left(c_1\int_0^u (u-y_{k+1})g_\theta(y_{k+1}\,|\,k)dy_{k+1}\right.$$
$$\left.+c_2\int_u^\infty (y_{k+1}-u)g_\theta(y_{k+1}\,|\,k)dy_{k+1}\right). \quad (55)$$

As shown in Section C, the optimum value u^* must satisfy (29). Because c is constant, the minimum value of $E_\theta\{\overline{C}(u)\}$ must also occur at u^*. In Fig. 1,

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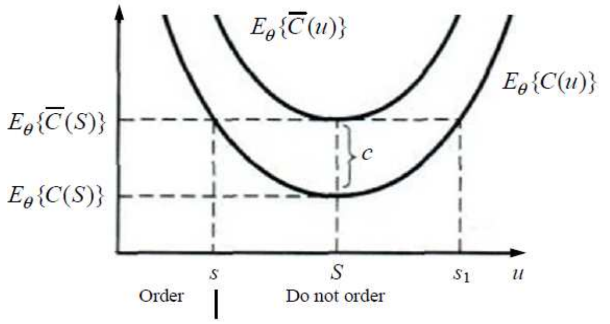

Fig. 1. (s–S) optimal ordering policy in a single-period model with setup cost.

$S = u^*$, and the value of s ($< S$) is determined from the equation

$$E_\theta\{C(s)\} = E_\theta\{\overline{C}(S)\} = c + E_\theta\{C(S)\}, \quad s < S. \tag{56}$$

The equation yields another value s_1 ($> S$), which is discarded. Assume that q is the amount on hand before an order is placed. How much should be ordered? This question is answered under three conditions: 1) $q < s$; 2) $s \le q \le S$; 3) $q > S$.

Case 1 *($q < s$)*. Because q is already on hand, its equivalent cost is given by $E_\theta\{C(q)\}$. If any additional amount $u - q$ ($u > q$) is ordered, the corresponding cost given u is $E_\theta\{\overline{C}(u)\}$, which includes the setup cost c. From Fig. 1, we have

$$\min_{u>q} E_\theta\{\overline{C}(u)\} = E_\theta\{\overline{C}(S)\} < E_\theta\{C(q)\}. \tag{57}$$

Thus, the optimal inventory policy in this case is to order $S - q$ units.

Case 2 *($s \le q \le S$)*. From Fig. 1, we have

$$E_\theta\{C(q)\} \le \min_{u>q} E_\theta\{\overline{C}(u)\} = E_\theta\{\overline{C}(S)\}. \tag{58}$$

Thus, it is not advantageous to order in this case and $u^* = q$.

Case 3 *($q > S$)*. From Fig. 1, we have for $u > q$,

$$E_\theta\{C(q)\} < E_\theta\{\overline{C}(u)\}. \tag{59}$$

This condition indicates that, as in case (2), is not advantageous to place an order – that is, $u^* = q$.

The optimal inventory policy, frequently referred to as the *$s - S$ policy*, is summarized as

$$\begin{aligned} &\text{If } x < S, \text{order } S - x, \\ &\text{If } x \ge s, \text{do not order.} \end{aligned} \tag{60}$$

The optimality of the $s - S$ policy is guaranteed because the associated cost function is convex.

E. Parametric Uncertainty

In the case when the parameter θ is unknown, the total expected prediction cost for the (k+1)th period,

$$E_{s_k}\{\overline{C}^{(1)}(u)\} = c + E_{s_k}\{C^{(1)}(u)\}$$

$$= c + \frac{k}{s_k}\left(c_1 \int_0^u (u - y_{k+1}) g^\bullet_{s_k}(y_{k+1} \mid k) dy_{k+1} \right.$$

$$\left. + c_2 \int_u^\infty (y_{k+1} - u) g^\bullet_{s_k}(y_{k+1} \mid k) dy_{k+1} \right), \tag{61}$$

is considered in the same manner as above.

IV. Conclusion

In this paper, we develop a new frequentist approach to improve predictive statistical decisions for inventory control problems under parametric uncertainty of the underlying distributions for the cumulative customer demand. Frequentist probability interpretations of the methods considered are clear. Bayesian methods are not considered here. We note, however, that, although subjective Bayesian prediction has a clear personal probability interpretation, it is not generally clear how this should be applied to non-personal prediction or decisions. Objective Bayesian methods, on the other hand, do not have clear probability interpretations in finite samples. For constructing the improved statistical decisions, a new technique of invariant embedding of sample statistics in a performance index is proposed. This technique represents a simple and computationally attractive statistical method based on the constructive use of the invariance principle in mathematical statistics.

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