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Approximation Methods Preserving Cones of Generalized Convex Functions

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***Abstract*—The main goal of the paper is to present constructions of linear approximation methods preserving the shape in the sense of cones of generalized convex functions and to examine their approximation properties using Korovkin type results for conservative linear approximation.**

***Index Terms*—shape-preserving approximation; linear approximation; degree of approximation**

I. Introduction

Different applications of computer-aided geometric design require to approximate functions with preservation of such properties as monotonicity, convexity, concavity and the like. The part of approximation theory that deals with this type of problem is known as the theory of shape preserving approximation. Over the past 30 years extensive study in the theory of shape-preserving approximation has brought about new results, the most substantial of which were outlined in [1], [2] and [3].

One of the main directions of research in the theory of shape-preserving approximation is the study of shape-preserving properties of Bernstein-type polynomials. It was shown by J. Pál [4] in 1925 that any convex function defined on [0,1] can be uniformly approximated by a sequence of convex algebraic polynomials on [0,1]. Some years later T. Popoviciu [5] proved that if f is k-monotone on [0,1], then Bernstein polynomial

$$B_n f(x) := \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right) \qquad (1)$$

also is monotone of order k on [0,1]. The papers [6], [7], [8], [9], [10] investigate the shape preserving and convergence properties of sequences of linear Bernstein-type operators. On the other hand, it is well-known that one of the shortcomings for Bernstein-type approximation is the low order of approximation [11].

The papers [12], [13] present the example of linear operator of finite rank n that preserves k-monotonicity and uses k-th derivative's values of approximated function at equidistant knots on [0,1], with optimal order of approximation n^{-2}. It should be noted that non-linear approximation methods preserving k-monotonicity are much better in the terms of approximation error than linear ones [14]. On the other hand, for sequences of linear operators preserving k-monotonicity (as well as intersections of cones) there are [15], [16], [17] simple convergence conditions (Korovkin type results).

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Definition 1. *Let $k \in \mathbb{N}$ and let $u_l \in C^{k-1}[0,1]$, $l = 0, \dots, k-1$, be such that a system $\{u_0, \dots, u_{k-1}\}$ is an extended complete Tchebycheff system on $[0,1]$ (ECT-system). We recall that a function f, defined on [0,1], is said to be convex relative to the system $\{u_0, \dots, u_{k-1}\}$, if*

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \dots & u_0(t_k) \\ \dots & \dots & \dots & \dots \\ u_{k-1}(t_0) & u_{k-1}(t_1) & \dots & u_{k-1}(t_k) \\ f(t_0) & f(t_1) & \dots & f(t_k) \end{vmatrix} \geq 0$$

for all choices of points $0 < t_0 < t_1 < \dots < t_k < 1$.

Definition 2. *Let*

$$C(u_0, \dots, u_{k-1})$$

denote the cone of all real-valued function defined on [0,1] and convex relative to the system $\{u_0, \dots, u_{k-1}\}$. In the following we will use the notation

$$V_k := C(u_0, \dots, u_{k-1})$$

for brevity.

In particular, if $u_0(x) = 1$, then $C(u_0)$ is the cone of all non-decreasing functions on $(0,1)$. If $u_0 = 1$, $u_1(x) = x$, then $C(u_0, u_1)$ is the cone of all convex functions on $(0,1)$. The review of the theory of generalized convex functions can be found in the book [18].

If function $f \in C[0,1]$ has shape properties, it usually means that element f belongs to a cone V in $C[0,1]$.

Definition 3. *Let V be a cone in $C[0,1]$. It is said that a linear operator $L : C[0,1] \to C[0,1]$ preserves the shape in the sense of the cone V if $L(V) \subset V$.*

The main goal of the paper is to present constructions of linear finite-dimensional methods preserving shape-property in the sense of the cone V_k and to examine their approximation properties.

II. Korovkin-type theorems

One of the most well-known classes of linear operators which preserve shape is the class of linear positive operators. Let V_0 denote the cone of all non-negative continuous functions defined on [0,1], i.e $V_0 := \{f \in C[0,1] \; : \; f \geq 0\}$.

Definition 4. *Recall that an operator L defined in $C[0,1]$ with range in $C[0,1]$, is called positive operator, if $L(V_0) \subset V_0$, i.e. if L preserves the shape in the sense of cone V_0.*

Results of Korovkin are classical in the theory of positive operators. Korovkin found [19] conditions of convergence of a sequence of linear positive operators to identity operator I in $C[0,1]$.

Let $e_j(t) = t^j$, $j = 0, 1, \ldots$. Let $\|\cdot\|$ denote uniform norm, $\|f\| = \sup_{t\in[0,1]} |f(t)|$.

Theorem 1. *Let $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1) $L_n(V_0) \subset V_0$, $n \geq 1$,
2) $\lim_{n\to\infty} \|(L_n - I)e_j\| = 0$, $j = 0, 1, 2$,

then

$$\lim_{n\to\infty} \|(L_n - I)f\| = 0$$

for all $f \in C[0,1]$.

In this section we use some results of [16] and prove Korovkin-type results for sequences of linear operators preserving shape-property in the sense of the cone $V_{h,k}(\sigma)$.

We need the following result proved in [16].

Let $R^{[0,1]}$ be the space of all real-valued functions defined on [0,1]. Let B be a subset of $R^{[0,1]}$, and A be subspace of $C[0,1]$ with $A \subset B$. Let $L : B \to R^{[0,1]}$ be a linear operator satisfying $L(A) \subset C[0,1]$.

Lemma 2. *Let $P = \{f \in B : Lf \geq 0\}$ and let V be a cone of A. Let U be a finite-dimensional subspace of A satisfying the following properties:*

1) *there exists Chebyshev system $\{w_0, \ldots, w_r\}$, $r \geq 2$, such that $L(U) = \text{span}\ \{w_0, \ldots, w_r\}$*
2) *for every point $z \in [0,1]$, there exists $\phi_z \in V \cap U$ such that*
   - *$L\phi_z(z) = 0 < L\phi_z(x)$ for all $x \in [0,1] \setminus z$*
   - *$\forall f \in A$, $\exists \alpha = \alpha(f) > 0$: $\beta > \alpha \Rightarrow \beta\phi_z + f \in V$*

Let $\{K_n\}_{n\geq1}$, $K_n : A \to B$, be a sequence of linear operators satisfying the following properies

1) *$K_n(P \cap V) \subset P$ for $n \geq 1$*
2) *for every $f \in U$, $L(K_n f)$ converges uniformly to Lf as $n \to \infty$*

Then for every $f \in A$, $L(K_n f)$ converges uniformly to Lf as $n \to \infty$

Let $u_l \in C^{k+2}[0,1]$, $l = 0, \ldots, k+2$, be such that a system $\{u_0, \ldots, u_{k+2}\}$ is an ECT-system on $[0,1]$.

Without loss of generality we can assume that functions $u_0, \ldots, u_{k+2} \in C^{k+2}[0,1]$ satisfy initial conditions $u_l^{(p)} = 0$, $p = 0, \ldots, l-1$, $l = 1, \ldots, k+2$. It is known [18] that the system $\{u_0, \ldots, u_{k+2}\}$ can be represented as

$$u_0(t) = \omega_0(t),$$

$$u_l(t) = \omega_0(t)\times \int_0^t \omega_1(\zeta_1) \int_0^{\zeta_1} \omega_2(\zeta_2) \ldots \int_0^{\zeta_{l-1}} \omega_l(\zeta_l) d\zeta_l \ldots d\zeta_1, \quad l = 1, \ldots, k+2,$$

where $\omega_0, \ldots, \omega_{k+2}$ are strictly positive functions on [0,1], such that $\omega_l \in C^{k-l}[0,1]$, $l = 0, \ldots, k+2$.

Let D_j, $j = 0, \ldots, k$, denote the first order differential operator

$$(D_j f)(t) = \frac{d}{dt}\left(\frac{f(t)}{\omega_j(t)}\right).$$

Denote $D^{[r]} = D_{r-1} \ldots D_0$, $r = 1, \ldots, k$, $D^{[0]} := I$. It is known [18] that

$$D^{[j+1]} u_{j+1} = \omega_{j+1},\ j = 0, \ldots, k-1,$$

$$D^{[j+1]} u_j = 0,\ j = 0, \ldots, k.$$

Note that $u_j \in V_j := C(u_0, \ldots, u_{j-1})$, $j = 1, \ldots, k+2$.

One of the most well-known examples for ECT-system is $\{e_0, e_1, \ldots, e_{k+2}\}$, i.e the system of monomial functions. The system is generated by $\omega_0 = 1$, $\omega_j = j$, $j = 1, \ldots, k+2$. Then $u_j = e_j$, $j = 0, \ldots, k+2$, and $D^{[j]} = D^j$, where D^j is the j-th differential operator, $D^j f(t) = \frac{d^j f(t)}{dt^j}$.

Theorem 3. *Let $L_n : C^k[0,1] \to C^k[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1) $L_n(V_k) \subset V_k$,
2) $\lim_{n\to\infty} \|(D^{[k]}u_j - D^{[k]}(L_n u_j)\| = 0$, $j = k, k+1, k+2$,

then

$$\lim_{n\to\infty} \|(D^{[k]}f - D^{[k]}(L_n f)\| = 0$$

for all $f \in C^k[0,1]$.

We will prove the following more general result.

Let $\sigma = (\sigma_0, \ldots, \sigma_k) \in R^{k+1}$, $\sigma_i \in \{-1, 0, 1\}$, and let h, k be two integers, such that $\sigma_h\sigma_k \neq 0$. Consider the cone

$$V_{h,k}(\sigma) = \bigcap_{l=h,\ \sigma_l\neq0}^{k} \sigma_l V_l. \tag{2}$$

Denote $\sigma^{[j]} = \{\sigma_i^{[j]}\}_{i\geq0}$, with $\sigma_i^{[j]} = 0$ for $i \neq j$ and $\sigma_j^{[j]} = \sigma_j$. Let $\sigma^{(j)} = \{\sigma_i^{(j)}\}_{i\geq0}$, with $\sigma_i^{(j)} = \sigma_i$ for $i \neq j$ and $\sigma_j^{(j)} = 0$.

Theorem 4. *Let $V_{h,k}(\sigma)$ be the cone defined in* (2)*. Let $L_n : C^k[0,1] \to C^k[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1) $L_n(V_{h,k}(\sigma)) \subset V_{h,k}(\sigma^{[k]})$,
2) $\lim_{n\to\infty} \|(D^{[k]}u_j - D^{[k]}(L_n u_j)\| = 0$, $j = h, \ldots, k+2$,

then

$$\lim_{n\to\infty} \|(D^{[k]}f - D^{[k]}(L_n f)\| = 0$$

for all $f \in C^k[0,1]$.

The proposition of Theorem 4 follows from Lemma 2 and Lemma 5.

Lemma 5. *For every point $z \in [0,1]$, there exists $\varphi_z \in$* span$\{u_h, \ldots, u_{k+2}\}$ *such that*

1) *$\varphi_z \in V_{h,k}(\sigma^{(k)})$;*
2) *$D^{[k]}\varphi_z(z) = 0 < D^{[k]}\varphi_z(x)$ for all $x \in [0,1] \setminus \{z\}$;*
3) *for every $f \in C^k[0,1]$ there exists $\alpha = \alpha(f) \geq 0$ such that for all $\beta > \alpha$ $\beta\varphi_z + f \in V_{h,k}(\sigma)$ holds.*

Proof: Take $z \in [0,1]$. Since the system $\{D^{[k]}u_k, D^{[k]}u_{k+1}, D^{[k]}u_{k+2}\}$ is a Tchebycheff system it is possible to choose $a_i = a_i(z) \in \mathbb{R}$, $i = 0, 1, 2$, such that

$$\sum_{i=0}^{2} a_i(z) D^{[k]} u_{k+i}(z) = 0 < \sum_{i=0}^{2} a_i(z) D^{[k]} u_{k+i}(x) \text{ for all } x \in [0,1] \setminus z.$$

Define a function $\varphi_z \in \text{span}\{u_0, \ldots, u_{k+2}\}$ in the following way

1) $D^{[k]}\varphi_z = \sigma_k \sum_{i=0}^{2} a_i(z) D^{[k]} u_{k+i}$;
2) for $p = h, \ldots, k-1$ we take $D^{[p]}\phi_z(0) = \sigma_p(1 + \beta_p)$, $\beta_p := \|\omega_{p+1} D^{[p+1]}\phi_z\|$.

It can be checked that $\phi_z \in V_{h,k}(\sigma^{(k)})$ and $\sigma_p D^{[p]}\varphi_z \geq 1$, $p = h, \ldots, k-1$. Indeed, let $\sigma_p \neq 0$, $h \leq p \leq k-1$. We have $\sigma_p D^{[p]}\phi_z(x) = \sigma_p D^{[p]}\phi_z(0) + \sigma_p \int_0^x (\omega_{p+1} D^{[p+1]}\phi_z(x)) \geq \sigma_p D^{[p]}\phi_z(0) - \beta_p \geq 1$ for all $x \in [0,1]$. ■

In paper of F. J. Muñoz-Delgado, V. Ramírez-González and D. Cárdenas-Morales [16] the following cone

$$C_{h,k}(\sigma) = \{f \in C^k[0,1] \ : \ \sigma_i D^i f \geq 0, \ h \leq i \leq k\}.$$

was considered. They proved [16] the next Korovkin-type result for sequences of linear operators preserving shape.

Theorem 6. *Let $L_n : C^k[0,1] \to C^k[0,1]$, $n \geq 1$, be a sequence of linear operators. If*

1) $L_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{[k]})$,
2) $\lim_{n\to\infty} \|D^k e_j - D^k(L_n e_j)\| = 0$, $j = h, \ldots, k+2$,

Then

$$\lim_{n\to\infty} \|D^k f - D^k(L_n f)\| = 0$$

for all $f \in C^k[0,1]$.

The result of Theorem 6 is a particular case of the result of Theorem 4 with $u_j = e_j$, $j = 0, \ldots, k+2$. It arises from the following facts:

1) the system $e_0, \ldots, e_{k+2}$ is extended complete Tchebycheff system on $[0,1]$;
2) e_j is a convex relative to the system $e_0, \ldots, e_{j-1}$, $j = h+1, \ldots, k+2$.

III. Preservation of Cones of Generalized Convex Functions

A. Properties of Generalized Polynomials

Let $p \in \mathbb{N} \cup \{0\}$, $u_l \in C^p[0,1]$, $l = 0, \ldots, p$, and let $\{u_0, \ldots, u_p\}$ be an ECT–system on $[0,1]$.

Definition 5. *Let*

$$L_p f(\,\cdot\,; y_0, y_1, \ldots, y_p) \in \operatorname{span}\{u_0, \ldots, u_p\}$$

denote the generalized polynomial which interpolates $f \in C[0.1]$ at points $0 \leq y_0 < y_1 < \ldots < y_p < 1$:

$$L_p f(y_i; y_0, y_1, \ldots, y_p) = f(y_i), \ i = 0, \ldots, p. \tag{3}$$

Denote $y_{-1} = -\infty$, $y_{p+1} = +\infty$.

Let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence with $\sigma_i \in \{-1, 0, 1\}$.

Lemma 7. *Let $f \in V_{0,p+1}(\sigma)$.*

1) *If $\sigma_0\sigma_{p+1} > 0$, then*
$$\sigma_0 L_p f(x; y_0, \ldots, y_p) \geq 0 \tag{4}$$
for all $x \in \bigcup_{i=0}^{[(p)/2]} [y_{p-(2i+1)}, y_{p-2i}]$.
2) *If $\sigma_0\sigma_{p+1} < 0$, then the inequality* (4) *holds for all $x \in \bigcup_{i=-1}^{[(p+1-2)/2]} [y_{p-(2i+2)}, y_{p-(2i+1)}]$.*

Proof: Suppose that $x \in (y_{l-1}, y_l)$, $l = 0, \ldots, p+1$. It follows from $f \in V_{0,p+1}(\sigma)$ that $\sigma_{p+1}\Delta_p f(x; y_0, \ldots, y_p) \geq 0$, where

$$\Delta_p f(x; y_0, \ldots, y_p) = (-1)^l \begin{vmatrix} u_0(x) & u_0(y_0) & \ldots & u_0(y_p) \\ \cdots & \cdots & \cdots & \cdots \\ u_p(x) & u_p(y_0) & \ldots & u_p(y_p) \\ f(x) & f(y_0) & \ldots & f(y_p) \end{vmatrix}.$$

It follows from

$$\Delta_p f(x; y_0, \ldots, y_p) = (-1)^{p+l}(L_p f(x; y_0, \ldots, y_p) - f(x)) \det(u_i(y_j))_{i=0,\ldots,p}^{j=0,\ldots,p}, \tag{5}$$

that $\sigma_{p+1}(-1)^{p+l} L_p f(x; y_0, \ldots, y_p) \geq \sigma_{p+1}(-1)^{p+l} f(x)$. Since $\sigma_0 f \geq 0$, the inequality (4) holds for appropriate x. ■

In the case $u_j = e_j$ Lemma 7 was proved in [20].

Lemma 8. *If $0 \leq y_0 < y_1 < \ldots < y_p < 1$, then*

$$L_p u_i(\,\cdot\,; y_0, \ldots, y_p) = u_i, \quad i = 0, \ldots, p,$$

where L_p is defined in Definition 5.

Lemma 9. *Let $p \in \mathbb{N} \cup \{0\}$ and let $\{u_0, \ldots, u_{p+1}\}$ be an ECT–system on $[0,1]$. Let $x \in (0,1)$ and $\{Y_n\}_{n\geq 1} := \{0 \leq y_{0,n} < y_{1,n} < \ldots < y_{p,n} < 1\}_{n\geq 1}$ be such that*

1) $\max_{1\leq i\leq p} |y_{i,n} - y_{i-1,n}| \leq \frac{1}{n}$*;*
2) *for some $1 \leq j \leq p$ the inclusions $x \in (y_{j-1,n}, y_{j,n})$ holds for all $n \geq 1$.*

Then

$$\lim_{n\to\infty} \|L_p u_{p+1}(x; y_0, \ldots, y_p) - u_{p+1}(x)\|_{C[y_{j-1,n}, y_{j,n}]} = 0.$$

B. Constructions of Linear Methods that Preserve Cones of Generalized Convex Functions

Let $p \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $0 \leq y_0 < y_1 < \ldots < y_p < 1$, and let $\{u_0, \ldots, u_{s+p}\}$ be an ECT–system on $[0,1]$. Let $f \in C^s[0,1]$ and denote $L_{s,p} f(\,\cdot\,; y_0, \ldots, y_p) \in \operatorname{span}\{u_0, \ldots, u_{s+p}\}$ the generalized polynomial uniquely defined by

1) $D^{[s]} L_{s,p} f(x; y_0, \ldots, y_p) = L_p(D^{[s]} f)(x; y_0, \ldots, y_p)$;
2) $D^{[i]} L_{s,p} f(0; y_0, \ldots, y_p) = 0$, $\quad i = 0, \ldots, s-1$, if $r > 0$.

If $\{u_0, \ldots, u_k\}$ is an ECT–system on $[0,1]$, then for any $0 \leq i \leq k$, $z \in [0,1]$, $g \in C^k[0,1]$ there is a unique function $F_{i,z}[g] \in \operatorname{span}\{u_0, \ldots, u_i\}$ such that

$$D^{[l]} F_{i,z}[f](z) = D^{[l]} g(z), \ l = 0, \ldots, i.$$

Let $n \in \mathbb{N}$, $x_i = i/n$, $\quad i = 0, \ldots, n$.

Let the linear operator $M_{k,n} : \ C^k[0,1] \to C^k[0,1]$ be defined in steps from left to right in the following way:

$$M_{k,n} f(x) = F_{k-1,x_0}[f - L_{k,2} f(\,\cdot\,; x_0, x_1)](x) + L_{k,2} f(x; x_0, x_1), \ x \in [x_0, x_1]\,; \tag{6}$$

$$M_{k,n} f(x) = F_{k-1,x_{j-1}}[M_{k,n} f - L_{k,2} f(\,\cdot\,; x_{j-1}, x_j)](x) + L_{k,2} f(x; x_{j-1}, x_j), \ x \in (x_{j-1}, x_j]\,, \ j = 2, \ldots, n. \tag{7}$$

Theorem 10. *Let $M_{k,n} : \ C^k[0,1] \to C^k[0,1]$ be the linear operator defined by* (6) *and* (7)*. Then*

1) $M_{k,n}(V_k) \subset V_k$*;*
2) *for any $f \in C^k[0,1]$*
$$\lim_{n\to\infty} \|D^{[k]} M_{k,n} f - D^{[k]} f\| = 0.$$

Proof: The proposition 1 of Theorem follows from Lemma 7. It follows from Lemma 8 that

$$D^{[k]} M_{k,n} u_i = D^{[k]} u_i, \ i = 0, \ldots, k+1. \tag{8}$$

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It follows from Lemma 9 that

$$\lim_{n\to\infty} \|D^{[k]}M_{k,n}u_{k+2} - D^{[k]}u_{k+2}\| = 0. \quad (9)$$

Finally, the proposition 2 of Theorem is the corollary of (8)–(9) and Theorem 3. ∎

C. The Example of Preservation k-Monotonocity

Denote $e_j(x) = x^j$, $j = 1, 2, \ldots$. In this section we consider the case $u_j = e_j$, $j = 0, 1, 2, \ldots$, i.e. $C(1, e_0, \ldots, e_{k-1})$ is the cone of all function f defined on [0,1] and convex relative to the system $e_0, \ldots, e_{k-1}$. Then $C(1, e_0, \ldots, e_{k-1})$ is the cone of all k-monotone functions defined on [0,1].

We will use the notation $\Delta^k := C(1, e_0, \ldots, e_{k-1})$. If f is a real-valued and k-times continuously differentiable function defined on $[0, 1]$, then $f \in \Delta^k$ iff $f^{(k)}(t) \geq 0$, $t \in [0, 1]$.

Denote by $C^k[0, 1]$, $k \geq 0$, the space of all real-valued and k-times continuously differentiable functions defined on $[0, 1]$, equipped with the norm

$$\|f\|_{C^k[0,1]} = \sum_{0\leq i\leq k} \frac{1}{i!} \sup_{x\in[0,1]} |D^i f(x)|, \quad (10)$$

where D^i denotes the i-th differential operator, $D^i f(x) = d^i f(x)/dx^i$, and $D^0 = I$ is the identity operator, and the derivatives are taken from the right at 0 and from the left at 1. If $f \in C^k[0, 1]$, then $f \in \Delta^k$ iff $f^{(k)}(t) \geq 0$, $t \in [0, 1]$.

It is said that a linear operator L of $C[0, 1]$ into $C[0, 1]$ preserves k-monotonicity, if $L(\Delta^k) \subset \Delta^k$.

Denote by $B^k[0, 1]$, $k \geq 0$, the space of all real-valued functions, whose k-th derivative is bounded on $[0, 1]$ endowed with the sup-norm (10).

Let $W^{(k+2)}_\infty[0, 1]$ be the Sobolev space of all real-valued, $(k + 1)$-times differentiable functions whose derivative of order $(k + 1)$ is absolutely continuous and whose derivative of order $k + 2$ is in $L^\infty[0, 1]$, $\|f\|_\infty := \text{ess sup}_{x\in[0,1]}|f(x)|$. Denote $B^{(k+2)}_\infty := \{f \in W^{(k+2)}_\infty[0, 1] : \|D^{k+2}f\|_\infty \leq 1\}$.

In the case $u_j = e_j$, $j = 0, \ldots, k + 2$, the operator $M_{k,n}$ defined in (6) and (7) can be presented as follows.

Let $k, n \in \mathbb{N}$, $n \geq k + 2$, $x_j = j/n$, $j = 0, 1, \ldots, n$, and let $\Lambda_{k,n} : C^k[0, 1] \to C^k[0, 1]$ be the linear operator defined in steps from left to right by (see also [12])

$$\Lambda_{k,n}f(x) = \sum_{l=0}^{k-1} \frac{1}{l!}x^l \left(D^l f(x_0) + \frac{(-1)^{k+1-l}}{(k + 1 - l)!n^{k-l}} D^k f(x_0)\right) + \frac{n}{(k + 1)!} \left[(x - x_0)^{k+1} D^k f(x_1) + (-1)^k (x_1 - x)^{k+1} D^k f(x_0)\right],\ x \in [0, x_1], \quad (11)$$

$$\Lambda_{k,n}f(x) = \sum_{l=0}^{k-1} \frac{1}{l!}(x - x_i)^l \left(D^l \Lambda_{k,n}f(x_i) + \frac{(-1)^{k+1-l}}{(k + 1 - l)!n^{k-l}} D^k f(x_i)\right) + \frac{n}{(k + 1)!} \left[(x - x_i)^{k+1} D^k f(x_{i+1}) + (-1)^k (x_{i+1} - x)^{k+1} D^k f(x_i)\right],$$
$$x \in (x_i, x_{i+1}],\quad i = 1, 2, \ldots, n - 1. \quad (12)$$

Theorem 11. *$\Lambda_{k,n} : C^k[0, 1] \to C^k[0, 1]$ is a continuous linear operator of finite rank $n + 1$, such that*

1) *$\Lambda_{k,n}(\Delta^k) \subset \Delta^k$;*
2) *for any $f \in C^k[0, 1]$*

$$\lim_{n\to\infty} \|D^k(\Lambda_{k,n}f) - D^k f\| = 0;$$

3) *there exists a constant $0 < c \leq 2^{-3}$ not depending on n such that*

$$\sup_{f\in B^{k+2}_\infty} \|D^k\Lambda_{k,n}f - D^k f\| \leq cn^{-2}. \quad (13)$$

Proof: The proposition 1 of Theorem is the corollary of Theorem 10, part 1.

The proposition 2 of Theorem follows from Theorem 10, part 2.

Let f be a function from $B^{(k+2)}_\infty$. Let $x \in [x_j, x_{j+1}]$. Then $D^k f \in W^{(2)}_\infty[0, 1]$ can be represented as

$$D^k f(x) = D^k f(x_j) + \frac{D^{k+1} f(x_j)}{1!} (x - x_j) + \int_{x_j}^1 (x - t)_+ D^{k+2} f(t)dt. \quad (14)$$

where $y_+ := \max\{y, 0\}$.

Note that $D^k(\Lambda_{k,n}f)$ is a piecewise linear function on [0,1] with the set of breakpoints $\{(x_j, D^k f(x_j))\}_{j=0,\ldots,n}$. If $x \in [x_j, x_{j+1}]$ then

$$D^k(\Lambda_{k,n}f)(x) = D^k(\Lambda_{k,n}f)(x_j) + \frac{D^{k+1}_+\Lambda_{k,n}f(x_j)}{1!} (x - x_j) + \int_{x_j}^1 (x-t)_+ D^{k+2}\Lambda_{k,n}f(t)dt, \quad (15)$$

where $D^{k+1}_+\Lambda_{k,n}f(x_j)$ is the right-hand side derivative of $D^k\Lambda_{k,n}f$ at point x_j.

It follows from (14) and (15) that if $x \in [x_j, x_{j+1}]$ then

$$\left(D^k(\Lambda_{k,n}f) - D^k f\right)(x) = (x - x_j) \left[n \left(D^k f(x_{j+1}) - D^k f(x_j)\right) - D^{k+1} f(x_j)\right] - \int_{x_j}^1 (x - t)_+ D^{k+2} f(t)dt = \int_{x_j}^1 \left(n (x - x_j)(x_{j+1} - t)_+ - (x - t)_+\right) D^{k+2} f(t)dt.$$

Since $\|D^{k+2}f\|_\infty \leq 1$, we have

$$\sup_{x\in[x_j,x_j]} \left|D^k(\Lambda_{k,n}f)(x) - D^k f(x)\right| \leq \sup_{x\in[0,\frac{1}{n}]} \int_0^{\frac{1}{n}} \left|nx\left(\frac{1}{n} - t\right)_+ - (x - t)_+\right| dt \leq \sup_{x\in[0,\frac{1}{n}]} \frac{1}{2}x\left(\frac{1}{n} - x\right) = \frac{1}{8n^2}. \quad (16)$$

It follows from (16) that

$$\left|D^k(\Lambda_{k,n}f)(x) - D^k f(x)\right| \leq \frac{1}{8n^2}$$

for every $x \in [0, 1]$. ∎

Note that linear operator $\Lambda_{k,n}$ defined in (11)-(12) is the minimal shape-preserving projection [21] on the first interval $[0, \frac{1}{n}]$, and then it is smoothly extended to the next

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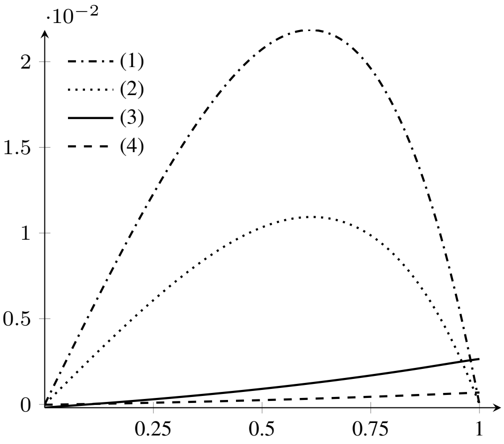

Figure 1. Errors of approximation of function $f(x) = \exp(x)$ on [0,1] by (1) Bernstein operator B_n, $n = 10$; (2) Bernstein operator B_n, $n = 20$; (3) operator $\Lambda_{k,n}$, $n = 10$, $k = 1$; (3) operator $\Lambda_{k,n}$, $n = 20$, $k = 1$

intervals. The paper [22] presents the example of linear finite-dimensional approximation method that preserves k-monotonicity of approximated functions and uses the values of function at equidistant points on [0,1] (rather than values of derivatives as it is in the definition of $\Lambda_{k,n}$). The example of linear linear operators preserving an intersection of cones can be found in the paper [23].

Figure 1 plots the comparison for errors of approximation of exponential function $f(x) = e^x$ on interval [0,1] by Bernstein operator B_n (defined in (1)) and operator $\Lambda_{k,n}$ for different n and $k = 1$. Line (1) of the plot is $B_{10}f - f$, line (2) of Figure 1 is the error $B_{20}f - f$, lines (3) and (4) plot the differences $\Lambda_{1,10}f - f$ and $\Lambda_{1,20}f - f$ respectively.

D. Applications

Approximation methods preserving cones have numerous applications in different areas of science and engineering such as computer graphics, numerical analysis, computational geometry, and many others industrial, medical, and scientific applications. In particular, software developers often need with mathematical and computational methods for the description of geometric objects as they arise in areas ranging from CAD/CAM to robotics and scientific visualization. Another application of shape-preserving algorithms is in the optimization theory and the theory of dynamic optimization. It is worth noting the paper [24] that presents algorithms for solving the dynamic programming problems based on shape-preserving methods of approximation and shows the applicability of the cone-preserving algorithms for the optimal growth problem.

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