Free Vibration Analysis of Moderately Thick, Sandwich, Circular Beams

Ü.N. Arıbaş, N. Eratlı, and M.H. Omurtag

***Abstract*—The objective of this study is to investigate parametrically the natural frequencies of the moderately thick, sandwich, circular beams. The element matrix is based on the Timoshenko beam theory including the rotary inertia in the formulation. The curved element involves two nodes and each node has three translations, three rotations, two shear forces, one axial force, two bending moment and one torque (12DOF). A parametric study is performed on the natural frequencies of sandwich beams with various thin facesheets. The results are verified with the available commercial CAD programs.**

***Index Terms*—composite beam, finite element, free vibration, Timoshenko beam theory**

I. Introduction

THE increased use of composites in many applications due to their attractive properties in strength, stiffness and lightness has resulted in a growing demand for engineers in the design of structures made of fiber-reinforced composite materials. Numerous texts dealing with the mechanics of composites by using various theories have been published to satisfy this demand [1-3].

The dynamic behaviors of the symmetrically or anti-symmetrically laminated straight or planar curved rods are investigated intensively in the literature. [4] investigated the natural frequencies and the Euler Buckling load of generally layered anisotropic laminated composite beams by parabolic shear deformation theory. [5] derived dynamic equations for the free vibration of generally layered composite beams using Hamilton's principle and the effects of rotary inertia and transverse shear are incorporated in the formulation. Analytical solutions are obtained by the method of the Lagrange multipliers. [6] studied the free vibration analysis of non-symmetric laminated cross-ply composite beams by including the coupling effects, shear deformation and rotary inertia based on Timoshenko beam theory. The numerical examples of composite beams are solved for the coupling effects, shear deformation and rotary inertia. [7] investigated the dynamic behavior of initially twisted laminated space rods under isothermal conditions based on Timoshenko beam theory by incorporating the Poisson effect, anisotropy of the material, rod curvature, rotary inertia and the shear and axial deformations are considered. [8,9] studied the free vibration analysis of symmetric cross-ply laminated circular arches by using the transfer matrix method. [10] investigated the dynamic analysis of symmetric cross-ply laminated beams based on a three-degree-of-freedom shear deformable beam theory by the Ritz method and Hamilton's principle under six different combinations of boundary conditions. The numerical results for different span ratios and lay-ups are obtained and compared with the results in the literature. [11] formulated a dynamic stiffness matrix that incorporate the Poisson's effect, couplings and rotary inertia for free vibration analysis of generally laminated composite beams with Hamilton's principle based on first-order shear deformation theory. The effects of the Poisson effect, material anisotropy, thickness ratio on the natural frequencies of the composite beams are investigated and the results are compared with the studies in the literature. [12] presented a formulation for the free vibration analysis of functionally graded spatial curved beams based on the first-order shear deformation theory by considering the effects of the thickness-curvature. Ritz method is used for the natural frequencies.

This work studies the natural frequencies of the moderately thick rectangular cross-sectional, sandwich, circular beams. For this purpose, a mixed finite element (FE) formulation comprising the Timoshenko beam theory is employed. As a numerical investigation, influence of the accurate torsional rigidity on the natural frequencies is investigated and results are verified by ANSYS 14.5.

II. The Constitutive Relations for Lamination

Letting the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$, the constitutive equation yields $\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}$ where $\mathbf{E}$ is the matrix of elastic constants. In order to derive the constitutive equations of a layered sandwich beam, firstly the assumptions made on stress, in accordance with beam geometry [7], secondly some reductions made on the constitutive relation of orthotropic materials for the three dimensional body by incorporating the Poisson's ratio [4].

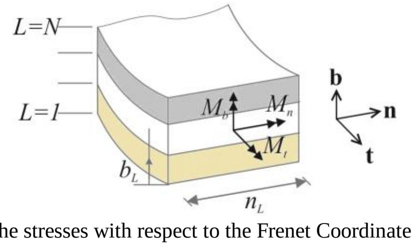

Fig. 1. The stresses with respect to the Frenet Coordinate System.
N : Total number of layers

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Ü.N. Arıbaş is with Istanbul Technical University, 34469, Istanbul, TURKEY (corresponding author to provide phone: 90-212-2856552; fax: 90-212-2856587; e-mail: umitaribas@hotmail.com).
N. Eratlı is with Istanbul Technical University, 34469, Istanbul, TURKEY (e-mail: eratli@itu.edu.tr).
M.H. Omurtag is with Istanbul Technical University, 34469, Istanbul, TURKEY (e-mail: omurtagm@itu.edu.tr).

In Frenet coordinate system (see Fig.1), paying attention to $\sigma_n = \sigma_b = \tau_{nb} = 0$, the constitutive relations yield

$$\begin{Bmatrix} \sigma_t \\ \tau_{bt} \\ \tau_{tn} \end{Bmatrix} = [\boldsymbol{\beta}] \begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} \quad (1)$$

where the 3×3 matrix $[\boldsymbol{\beta}]$ is the matrix in terms of the orthotropic material constants. Timoshenko beam theory requires shear correction factors and it is assumed to be $5/6$ for a general rectangular cross-section. By means of the kinematic equations

$$\begin{aligned} u_t^* &= u_t + b\,\Omega_n - n\,\Omega_b \\ u_n^* &= u_n - b\,\Omega_t \\ u_b^* &= u_b + n\,\Omega_t \end{aligned} \quad (2)$$

where, u_t^*, u_n^*, u_b^* are displacements at the beam continuum and u_t, u_n, u_b are displacements on the beam axis and Ω_t, Ω_n and Ω_b present the rotations of the beam cross-section around the *t*, *n* and *b* Frenet coordinates, respectively. The strains which are derived from (2)

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \end{Bmatrix} + b \begin{Bmatrix} \dfrac{\partial \Omega_n}{\partial t} \\ 0 \\ -\dfrac{\partial \Omega_t}{\partial t} \end{Bmatrix} + n \begin{Bmatrix} -\dfrac{\partial \Omega_b}{\partial t} \\ \dfrac{\partial \Omega_t}{\partial t} \\ 0 \end{Bmatrix} \quad (3)$$

and the constitutive equations for a single layer yield to the form

$$\begin{Bmatrix} \sigma_t \\ \tau_{bt} \\ \tau_{tn} \end{Bmatrix} = [\boldsymbol{\beta}] \left\{ \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \end{Bmatrix} + b \begin{Bmatrix} \dfrac{\partial \Omega_n}{\partial t} \\ 0 \\ -\dfrac{\partial \Omega_t}{\partial t} \end{Bmatrix} + n \begin{Bmatrix} -\dfrac{\partial \Omega_b}{\partial t} \\ \dfrac{\partial \Omega_t}{\partial t} \\ 0 \end{Bmatrix} \right\} \quad (4)$$

By obtaining strains for rod geometry due to displacements [12], the forces and moments for a layer can be derived by analytical integration of the stresses in each layer through the thickness of the cross-section, respectively.

$$T_t = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \quad (5)$$

$$T_b = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{bt} \mathrm{d}b \right) \mathrm{d}n \right) \quad (6)$$

$$T_n = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \quad (7)$$

$$\begin{aligned} M_t = &\sum_{L=1}^{N} \left(-\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\,\tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \\ &+ \sum_{L=1}^{N} \left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\,\tau_{tb} \mathrm{d}n \right) \mathrm{d}b \right) \end{aligned} \quad (8)$$

$$M_n = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \quad (9)$$

$$M_b = -\sum_{L=1}^{N} \left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\sigma_t \mathrm{d}n \right) \mathrm{d}b \right) \quad (10)$$

where, N is the number of the layer, n_L is the width of the layer, b_L and b_{L-1} are the directed distances to the bottom and the top of the L^{th} layer where *b* is positive upward. The constitutive equation in a matrix form:

$$\begin{Bmatrix} T_t \\ T_n \\ T_b \\ \hdashline M_t \\ M_n \\ M_b \end{Bmatrix} = \sum_{L=1}^{N} \left[\begin{array}{c:c} \mathbf{E}_T^L & \mathbf{E}_{TM}^L \\ \mathbf{E}_{MT}^L & \mathbf{E}_M^L \end{array} \right] \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \hdashline \dfrac{\partial \Omega_t}{\partial t} \\ \dfrac{\partial \Omega_n}{\partial t} \\ \dfrac{\partial \Omega_b}{\partial t} \end{Bmatrix} \quad (11)$$

or, since $[\mathbf{C}] = [\mathbf{E}]^{-1}$, in accordance with (3) and (4), (11) yields to the form

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{tn} \\ \gamma_{bt} \\ \hdashline \kappa_t \\ \kappa_n \\ \kappa_b \end{Bmatrix} = \left[\begin{array}{c:c} \mathbf{C}_T & \mathbf{C}_{TM} \\ \hdashline \mathbf{C}_{MT} & \mathbf{C}_M \end{array} \right] \begin{Bmatrix} T_t \\ T_n \\ T_b \\ \hdashline M_t \\ M_n \\ M_b \end{Bmatrix} \quad (12)$$

where $\kappa_t, \kappa_n, \kappa_b$ are curvatures.

III. THE FIELD EQUATIONS

The field equations for the isotropic homogenous spatial bar [13-15], which are based on the Timoshenko beam theory for orthotropic material are,

$$\left. \begin{aligned} -\mathbf{T}_{,s} - \mathbf{q} + \rho A \ddot{\mathbf{u}} = \mathbf{0} \\ -\mathbf{M}_{,s} - \mathbf{t} \times \mathbf{T} - \mathbf{m} + \rho \mathbf{I} \ddot{\boldsymbol{\Omega}} = \mathbf{0} \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} \mathbf{u}_{,s} + \mathbf{t} \times \boldsymbol{\Omega} - \mathbf{C}_T \mathbf{T} - \mathbf{C}_{TM} \mathbf{M} = \mathbf{0} \\ \boldsymbol{\Omega}_{,s} - \mathbf{C}_{MT} \mathbf{T} - \mathbf{C}_M \mathbf{M} = \mathbf{0} \end{aligned} \right\} \quad (14)$$

where $\mathbf{u}\,(u_t, u_n, u_b)$ is the displacement vector, $\boldsymbol{\Omega}\,(\Omega_t, \Omega_n, \Omega_b)$ is the cross section rotation vector. $\ddot{\mathbf{u}}$ and $\ddot{\boldsymbol{\Omega}}$ are the accelerations of the displacement and rotations,

$\mathbf{T}(T_t, T_n, T_b)$ defines the force vector, $\mathbf{M}(M_t, M_n, M_b)$ is the moment vector, ρ is the material density. A is the area of the cross section, $\mathbf{I}$ stores the moments of inertia, $\mathbf{C}_T$, $\mathbf{C}_M$, $\mathbf{C}_{TM} = \mathbf{C}_{MT}^T$ are compliance matrices where $\mathbf{C}_{TM}$, $\mathbf{C}_{MT}$ are coupling matrices [16]. $\mathbf{q}$ and $\mathbf{m}$ are the distributed external force and moment vectors, respectively. Once the motion is considered as harmonic for the free vibration of the beam, the conditions $\mathbf{q} = \mathbf{m} = \mathbf{0}$ are satisfied.

IV. THE FUNCTIONAL

Incorporating Gâteaux differential with potential operator concept [17] yields the functional in terms of (13)-(14)

$$\begin{aligned}\mathbf{I}(\mathbf{y}) = &-[\mathbf{u}, \mathbf{T}_{,s}] - [\mathbf{M}_{,s}, \mathbf{\Omega}] + [\mathbf{t} \times \mathbf{\Omega}, \mathbf{T}] - \tfrac{1}{2}\{[\mathbf{C}_T \mathbf{T}, \mathbf{T}] \\ &+ [\mathbf{C}_{TM} \mathbf{M}, \mathbf{T}] + [\mathbf{C}_{MT} \mathbf{T}, \mathbf{M}] + [\mathbf{C}_M \mathbf{M}, \mathbf{M}]\} \\ &- \tfrac{1}{2}\rho A \omega^2 [\mathbf{u}, \mathbf{u}] - \tfrac{1}{2}\rho\omega^2 [\mathbf{I}\,\mathbf{\Omega}, \mathbf{\Omega}] + \left[\left(\mathbf{T} - \hat{\mathbf{T}}\right), \mathbf{u}\right]_\sigma \\ &+ \left[\left(\mathbf{M} - \hat{\mathbf{M}}\right), \mathbf{\Omega}\right]_\sigma + [\hat{\mathbf{u}}, \mathbf{T}]_\varepsilon + \left[\hat{\mathbf{\Omega}}, \mathbf{M}\right]_\varepsilon\end{aligned} \tag{15}$$

which is original mixed finite element formulation for the literature. In (15), the square brackets indicate the inner product, the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively.

V. THE MIXED FINITE ELEMENT

The linear shape functions $\phi_i = (\varphi_j - \varphi)/\Delta\varphi$ and $\phi_j = (\varphi - \varphi_i)/\Delta\varphi$ are employed in the FE formulation where $\Delta\varphi = (\varphi_j - \varphi_i)$. i, j represent the node numbers of the curved element. φ_i and φ_j are the horizontal angle at i and j nodes, respectively. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element [14,15]. Calculation of the natural free vibration frequencies of a structural system yields to the following standard eigenvalue problem,

$$\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{\mathbf{u}\} = \{\mathbf{0}\} \tag{16}$$

where, $[\mathbf{K}]$ and $[\mathbf{M}]$ are the system and mass matrix of the entire domain, respectively. $\mathbf{u}$ is the eigenvector (mode shape) and ω depicts the natural angular frequency of the system.

VI. NUMERICAL EXAMPLES

In this study, the natural frequencies of the moderately thick rectangular cross-section, sandwich, circular beams with both ends clamped are analyzed by using the mixed FE algorithm based on Timoshenko beam theory. Firstly, the convergence analysis of this FE algorithm is performed and a comparison with the literature is studied. Next, a benchmark example is solved and the results are compared with the commercial FE program ANSYS. Torsional rigidity of an arbitrary composite cross-section requires special care. Therefore, the torsional rigidity for composite cross-sections GI_t is calculated by an FE solution based on Poisson's equation [15].

A. *The Convergence Analysis and Comparison*

A number of problems are solved on the fundamental natural frequencies of laminated $(0^o/90^o/0^o)$ circular beams. The material and geometrical properties are as follows: $E_1/E_2 = 40$, $E_3/E_2 = 1$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0.25$, the radius of circular beam is $R = 15\,\text{m}$, the opening angle is 90^o, the square cross-section ($b/h = 1$) is used where h is the thickness of the beam.

The circular beam with three different boundary conditions (fixed-fixed, fixed-free, fixed-simple) and two different $R/h = 5, 25$ ratios is solved by discretizing the beam using 40, 80 and 150 finite elements and the dimensionless fundamental frequency results which is obtained for 80 elements are compared with [9]. The torsional rigidity calculation given by [9] is an approximated formulation, thus it is denoted in Tables 1-3 by the notation $\approx GI_t$. The definition of non-dimensional frequency is

$$\bar{\omega} = \omega R^2 \sqrt{\frac{\rho}{E_2 h^2}} \tag{17}$$

The precision in determining the torsional rigidity of a composite cross section has a great importance since it has considerable influence on the natural frequencies.

TABLE 1
The dimensionless fundamental frequency for fixed-fixed boundary condition, N_e : number of elements

R/h	N_e	Mixed FE GI_t	Mixed FE $\approx GI_t$	[9]
5	40	4.958	5.885	5.94
	80	4.958	5.885	
	150	4.958	5.885	
25	40	7.031	10.73	11.08
	80	7.031	10.73	
	150	7.030	10.73	

TABLE 2
The dimensionless fundamental frequency for fixed-free boundary condition, N_e : number of elements

R/h	N_e	Mixed FE GI_t	Mixed FE $\approx GI_t$	[9]
5	80	0.412	0.840	0.909
25	80	0.417	0.866	0.946

TABLE 3
The dimensionless fundamental frequency for fixed-simple boundary condition, N_e : number of elements

R/h	N_e	Mixed FE GI_t	Mixed FE $\approx GI_t$	[9]
5	80	2.196	3.532	3.669
25	80	2.426	4.456	4.740

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B. The Benchmark Example

As a benchmark example, a composite circular beam having rectangular composite cross-section which is made of steel facesheets on the top and bottom with a concrete core as shown in Figure 1 is considered. This circular beam is fixed at both ends. The material properties and geometrical properties are as follows: the modulus of elasticity for the facesheets is $E_s = 210\,\text{GPa}$, Poisson's ratio is $\upsilon_s = 0.3$ and the material density is $\rho_s = 7850\,\text{kg/m}^3$. The modulus of elasticity for concrete $E_c = 30\,\text{GPa}$, Poisson's ratio is $\upsilon_c = 0.2$ and the material density is $\rho_c = 2400\,\text{kg/m}^3$. The radius of composite beam is $R = 12\,\text{m}$, the opening angle is 180^o. The rectangular cross-section with two different height ($h_1 = 0.8\,\text{m}$, $h_2 = 2.4\,\text{m}$) is employed where they have the same width $b = 0.4\,\text{m}$. The thickness of face sheets is $t = h/8$ for both rectangular cross-sections. Through the analysis, the first five natural frequencies of the composite circular beam are calculated using 80 mixed finite elements. The results are compared with ANSYS 14.5 using solid elements of a fine mesh configuration and presented in Tables 4-5.

As the thickness increases, an increasing trend is observed for the natural frequencies. It is observed that the effect of torsional rigidity of composite cross-sections is very imported on the calculation of natural frequencies.

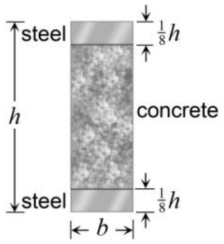

Figure 1. Composite rectangular cross-section

TABLE 4
The first five natural frequency (in Hz) for fixed-fixed boundary condition
% dif.: (Mixed FE -ANSYS)×100/ Mixed FE, $h_1 = 0.8\,\text{m}$

ω	Mixed FE		ANSYS
	GI_t	%dif.	
1	2.1818	-1.8	2.2215
2	2.4929	-0.2	2.4973
3	5.4731	-0.2	5.4827
4	6.4582	0.1	6.4538
5	10.150	-0.2	10.169

TABLE 5
The first five natural frequency (in Hz) for fixed-fixed boundary condition
% dif.: (Mixed FE -ANSYS)×100/ Mixed FE, $h_2 = 2.4\,\text{m}$

ω	Mixed FE		ANSYS
	GI_t	%dif.	
1	2.4929	-0.4	2.5041
2	3.5955	-11.7	4.0164
3	5.4731	-0.5	5.4984
4	10.150	-0.5	10.204
5	11.594	-1.2	11.735

VII. CONCLUSION

In this study, the effect of the torsional rigidity on the free vibration analysis of the moderately thick rectangular cross-section, sandwich, circular beams are investigated via mixed FE algorithm. This algorithm is based on the Timoshenko beam theory. The finite element formulation of the circular beam geometry is derived using the exact curvatures at the nodal points and their interpolations through the element axis. The accuracy of formulation and the influence of the torsional rigidity on the natural frequencies are discussed with the literature and verified with ANSYS 14.5. This study demonstrates that the calculation torsional rigidity is imported for a composite cross-section.

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