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Impulsive Differential Equations by Using R-K Method of Fourth Order

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***Abstract*—The impulsive differential equations represent more natural framework for mathematical modeling for many real life situations in the field of engineering, biology, chemistry, physics, control systems, population dynamics etc. as compared to the theory of ordinary differential equations. Still many impulsive differential equations cannot be solved analytically or it is very difficult to solve because the solution is not continuous at impulse moments. In this paper, an algorithm to solve impulsive differential equations by using R-K method of fourth order is presented. It is further elaborated with the help of numerical example and diagrams that the impulses do contribute to improve the accuracy of numerical solutions.**

***Index Terms*—Impulsive differential equations; Impulses at fixed moment; R-K method of fourth order; Numerical methods.**

I. Introduction

MANY evolution processes are characterized with the fact that at certain moments of time, they experience change of state abruptly. This is due to short term perturbations whose duration is negligible in comparison with the duration of the process. It is assumed naturally, that those perturbations act instantaneously in the form of impulses. Therefore impulsive differential equations represent more natural framework for mathematical modeling for many real life situations in the field of engineering, biology, chemistry, physics, control systems, population dynamics etc. as compared to the theory of ordinary differential equations. The pioneer papers in this theory were written by A. D. Myshkis and V. D. Milman in 1960s [11].In spite of its importance, many solutions in context to the impulsive differential equations are done analytically. Some of the famous researchers who presented significance results are V. Lakshmikantham, D. Bainov, P. Simeonov and many others [1], [2], [3], [4], [5], [6], [9], [10], [12].However many impulsive differential equations cannot be solved analytically or it is very difficult to solve them. Therefore, numerical solutions of impulsive differential equations plays very crucial role in such situations. In this paper, the numerical solutions of impulsive differential equations are obtained by using R-K method of fourth order. The algorithm proposed is interpreted according to the theory of impulsive differential equations written by V. Lakshmikantham et. al [12]. This method has improved the accuracy of solutions at each step and better numerical solution of the problem is obtained as compared to other numerical methods discussed in [7], [8].

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This paper is organized as follows. In section 2, we present some notations and definitions. In sections 3, numerical algorithm to find the solution of impulsive differential equations is proposed. Numerical example along with diagrams to compare results with analytic solution is given in section 4. At last we conclude our results conclusion section.

II. Impulsive Differential equations

Let R and Z be set of real and integer numbers respectively. Let $X = R^n$ and $T = \{t_k; k \in Z\} \subset R$ where $t_k < t_{k+1}$ for all $k \in Z, t_k \to \infty$ as $k \to \infty$. Also $t_k^+ = t_k + 0$ and $t_k^- = t_k - 0$.
if $\Omega \subset R$ is any real interval, we suppose that $x(t) = [x_1(t) \quad x_2(t)........x_n(t)]^T$ is a vector of unknown functions, and
$f(t, x) : \Omega \times X \to X$, such that $f(t, x) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), ...x_n(t)) \\ f_2(t, x_1(t), x_2(t), ...x_n(t)) \\ .... \\ .... \\ f_n(t, x_1(t), x_2(t), ...x_n(t)) \end{bmatrix}$ is continuous operator on every $[t_k, t_{k+1}] \times X$.

Definition 2.1: A system of differential equation of the form

$$\begin{aligned} x^{'}(t) &= f(t, x), \quad t \neq t_k \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad t = t_k \end{aligned} \tag{1}$$

where $I_k : X \to X$ are continuous operator at impulse moments, is called impulsive differential equation with initial conditions $x_{t_0} = x_0$.

Definition 2.2 : Any set of functions $\phi_i(t), i = 1, 2, 3....n$ is said to be a solution of impulsive differential equation (1) if it satisfies the system of differential equations along with conditions of jump and initial conditions.
A problem of existence and uniqueness of solutions of impulsive differential equation (1) is reduced to that corresponding ordinary differential equations i.e. $x^{'}(t) = f(t, x)$

Let $x(t)$ be the solution of impulsive differential equation (1) which satisfies the initial condition $x_{t_0} = x_0$.Also let Ω^+, Ω^- be maximal intervals on which the solution can be continued to the right and left of $t = t_0$ respectively. Then the next expression is valid:

$$x(t) = \begin{cases} x_0 + \int_{t_0}^{t} f(s, x(s)) + \sum_{t_0 < t_k < t} I_k(x(t_k)) & \text{for } t \in \Omega^+, \\ x_0 + \int_{t_0}^{t} f(s, x(s)) - \sum_{t_0 < t_k < t} I_k(x(t_k)) & \text{for } t \in \Omega^-. \end{cases} \tag{2}$$

III. Algorithm

Consider the impulsive differential equation (1) along with initial condition $x(t_0) = x_0$ be given. Let I_k be impulsive operators that act at the impulse moment t_k for all $k \in Z$ and which can be described by matrices of order $n \times n$ and $x(t)$ be the solution whose value is to be determine for fixed value of $t = t_s$ where $t_s > t_0$ i.e. $t_s \in \Omega^+$. Also we denote index of iteration with $i(i = 1, 2, 3, ...)$.Let h be the increment of t for each iteration. Now, algorithm consists of the following steps:
1. At the initial moment i.e. $t = t_0$ set $x = x_0$ given in initial condition.
2. Apply R-K method of fourth order to get the next values of $x(t)$ up to the first impulse moment as follows:

$$x^{(i+1)} = x^{(i)} + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = hf(t^{(i)}, x^{(i)})$$

$$K_2 = hf(t^{(i)} + \frac{h}{2}, x^{(i)} + \frac{h}{2}K_1)$$

$$K_3 = hf(t^{(i)} + \frac{h}{2}, x^{(i)} + \frac{h}{2}K_2)$$

$$K_4 = hf(t^{(i)} + h, x^{(i)} + hK_3)$$

3. At the impulse momenti.e. $t = t_k$ impulsive operator I_k acts and brings rapid changes in solution and it becomes $x(t_k) = x(t_k) + I_k(t_k)$.
4. Repeat the step 2 and step 3 till next impulse moment and then apply impulsive operator concerned with that particular impulse moment.
5. The above process is repeated until we encounter with the desired values i.e $x(t_s)$ is obtained.

IV. Numerical Example

Consider the impulsive differential equation

$$\begin{aligned} x^{'}(t) &= f(t,x), \quad t \neq t_k \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad t = t_k \\ x(t_0) &= x_0 \end{aligned} \tag{3}$$

and $t_0 = 0$

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad x_0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad f(t,x) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 \end{bmatrix}$$

Its impulsive operator at $t_k = 1$ is given by

$$I_1 = \begin{bmatrix} 1.0000 & 3.6565 \\ 0.0000 & -0.8020 \end{bmatrix}$$

Now, we have to approximate the value at $t = 1.9$
We applied the algorithm for R-K method of fourth order by taking step size $h = 0.1$ for each iteration . Then we compared the results obtained by using analytical expression for the solution. The equations of the analytic solution of (3) are as follows :

$$x(t) = \begin{cases} x_1(t) = -\frac{2}{3}e^{-t} + \frac{8}{3}e^{2t}, & \text{for } t \in (0,1] \\ x_2(t) = -\frac{2}{3}e^{-t} - \frac{4}{3}e^{2t}, & \text{for } t \in (0,1] \\ & \\ x_1(t) = -1.8136e^{-t} + 0.3606e^{2t}, & \text{for } t \in (1,2] \\ x_2(t) = -1.8136e^{-t} - 0.1803e^{2t}, & \text{for } t \in (1,2] \end{cases} \tag{4}$$

The numerical values of solution are shown in Table **??** are obtained by using MATLAB programming and results are compared with the results of analytic expression. Absolute errors are also given which is smaller at every step as compared to the absolute errors by using Euler method and Taylor series method as discussed in [7], [8].

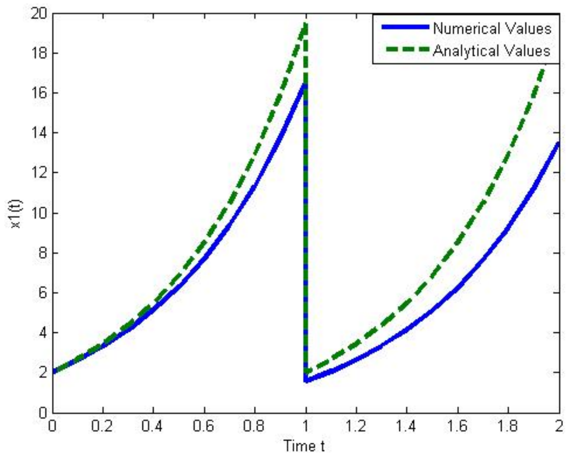

Fig. 1. Comparison of approximate values of $x_1(t)$ versus t ,between R-K method of fourth order and analytic method

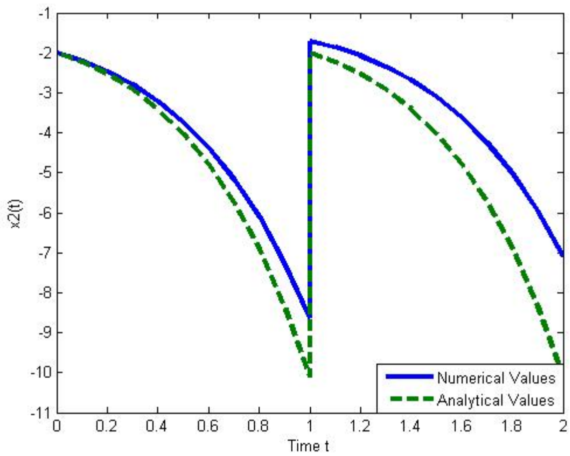

Fig. 2. Comparison of approximate values of $x_2(t)$ versus t ,between R-K method of fourth order and analytic method

Remark: It can be observed that absolute error goes on increasing before the impulse is encountered,then after the impulse moment it suddenly came down and numerical solution is very close to analytic solution.Therefore, accuracy

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of numerical solutions can be improved by increasing the frequency of impulses.

TABLE I
DETAILS OF CALCULATIONS BY R-K METHOD AND ANALYTIC METHOD

t	RK x1(t)	Anal. x1(t)	Error x1(t)	RK x2(t)	Anal. x2(t)	Error x2(t)
0	2	2	0	-2	-2	0
0.1	2.605036842	2.6538	0.048763158	-2.203016758	-2.2318	0.028783242
0.2	3.3122984	3.4324	0.1201016	-2.467046456	-2.5349	0.067853544
0.3	4.144425552	4.3651	0.220674448	-2.802424407	-2.9234	0.120975593
0.4	5.128434919	5.4879	0.359465081	-3.221771819	-3.4143	0.192528181
0.5	6.296622523	6.8444	0.547777477	-3.740437869	-4.0287	0.288262131
0.6	7.687648042	8.4878	0.800151958	-4.377033047	-4.7927	0.415666953
0.7	9.347836329	10.4828	1.134963671	-5.154072006	-5.738	0.583927994
0.8	11.33274027	12.9085	1.575759728	-6.098747872	-6.9036	0.804852128
0.9	13.70901793	15.8614	2.152382067	-7.243864399	-8.3372	1.093335601
1	16.55668761	19.4589	2.902212392	-8.628957723	-10.0973	1.468342277
1	1.5615913	1.997	0.4354087	-1.708533629	-1.9993	0.290766371
1.1	2.063538318	2.6507	0.587161682	-1.867195667	-2.2309	0.363704333
1.2	2.648329827	3.4287	0.780370173	-2.076465095	-2.5337	0.457234905
1.3	3.33454783	4.3608	1.02625217	-2.344718978	-2.9218	0.577081022
1.4	4.144344435	5.4827	1.338355565	-2.68221037	-3.4122	0.72998963
1.5	5.10418134	6.8382	1.73401866	-3.101429012	-4.0261	0.924670988
1.6	6.24571689	8.4803	2.23458311	-3.617536709	-4.7894	1.171863291
1.7	7.606870693	10.4737	2.866829307	-4.248892301	-5.7338	1.484907699
1.8	9.233101826	12.8975	3.664398174	-5.017684159	-6.8984	1.880715841
1.9	11.17894393	15.848	4.669056071	-5.950691786	-8.3309	2.380208214
2	13.50984921	19.4426	5.932750789	-7.080202455	-10.0895	3.009297545

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V. CONCLUSION

In this paper, we proposed a general numerical method for treating the impulsive differential equations with impulse effect at fixed moments. We intercepted the numerical algorithm following the theory of impulsive differential equations by using R-K method of fourth order. This type of work has not been done by researchers. Therefore, many studies have to be done in order to enhance and verify the existing results. We have compared our results with analytic results with the help of diagrams which have minimum absolute error as compared to other single step methods used in the literature. It is also discussed that by increasing the frequency of impulses the accuracy of numerical solutions can be improved.

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