A Handy Approximation Technique for Closed-form and Approximate Solutions of Time-Fractional Heat and Heat-Like Equations with Variable Coefficients

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***Abstract*— In this paper, we propose a handy approximation technique (HAT) for obtaining both closed-form and approximate solutions of time-fractional heat and heat-like equations with variable coefficients. The method is relatively recent, proposed via the modification of the classical Differential Transformation Method (DTM). It devises a simple scheme for solving the illustrative examples, and some similar PDEs. Besides being handy, the results obtained converge faster to their exact forms. This shows that this modified DTM (MDTM) is very efficient and reliable. It involves less computational work, even without given up accuracy. Therefore, we strongly recommend it for solving both linear and nonlinear time-fractional partial differential equations (PDEs) with applications in other aspects of pure and applied sciences, management, and finance.**

***Index Terms*— time-fractional differential equations; modified DTM; heat and heat-like equations; variable coefficients, closed-form solutions.**

I. Introduction

Many physical problems in various fields of pure and applied sciences are modelled mathematically by partial differential equations. Heat equations are special version of parabolic partial differential equations (PPDEs) governing heat diffusion and heat-like diffusive processes.

Heat equations are of great importance in diverse areas of sciences and engineering. It is highly linked to the study of Brownian motion through the application of the Fokker-Planck equation (application in probability theory) [1]. In financial mathematics, the heat equation can also be used for the solutions of financial models like the Black-Scholes option pricing model [2], integro-differential model [3] and so on.

In the sequel, the heat equation will be generalized to the time-fractional case (that is, of non-integer order). The study of fractional calculus has greatly attracted the attention of many researchers because of its suitability for the generalization of fractional differential equations [4].

Fractional differential equations are seen as alternatives to non-linear differential equations [4]. Many researchers have proposed, adopted and applied various methods in search for solutions of heat and heat-like equations, and related PDEs [5-14]. Recently, while Secer [15] applied DTM to heat-like equations, we hereby propose the modified DTM for less computational work among other merits.

In this work, a relatively new version of the modification referred to as modified differential transform method (MDTM) will be applied to heat and heat-like PDEs for exact and numerical solutions. It is noteworthy saying that the MDTM has advantages over the decomposition methods and the classical DTM as the computational time required is minimal, and for ease and simplicity of usage.

II. Fractional Calculus: Preliminaries and Notations

In fractional calculus, the power of the differential operator is considered a real or complex number. Hence, the following definitions [16-18]:

Definition 1: *Fractional derivative in gamma sense*

Let $D=\dfrac{d(\cdot)}{dx}$ and J be differential and integral operators respectively, with the gamma function of $h(x)$ being defined as:

$$\Gamma(n)=\int_0^\infty e^{-x}t^{n-1}dt,\ \operatorname{Re}(n)>0 \quad ,$$

$$\Gamma(n+1)=n!,\ \Gamma(1/2)=\sqrt{\pi} \qquad (1)$$

Equation (1) in terms of gamma sense is expressed as:

$$D^{\alpha}h(x)=\frac{d^{\alpha}h(x)}{dx^{\alpha}}=\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha} \qquad (2)$$

Equation (2) is referred to as a fractional derivative of $h(x)$, of order α, if $\alpha\in\mathbb{R}$.

Definition 2: Suppose $h(x)$ is defined for $x>0$, then:

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$$(Jh)(x)=\int_0^x h(s)ds \tag{3}$$

and as such, an arbitrary extension of (3) (i.e. Cauchy formula for repeated integration) yields:

$$(n-1)!\left(J^n h\right)(x)=\int_0^x (x-s)^{n-1}h(s)ds \tag{4}$$

While the gamma sense of (4) is:

$$\Gamma(\alpha)\left(J^\alpha h\right)(x)=\int_0^x (x-s)^{\alpha-1}h(s)ds\ ,\ \alpha>0,\ t>0. \tag{5}$$

Equation (5) is the Riemann-Liouville fractional integration of order α .

Definition 3: *Riemann-Liouville fractional derivative*

$$D^\alpha h(x)=\frac{d^\phi\left(J^{\phi-\alpha}h(x)\right)}{dx^\phi} \tag{6}$$

Definition 4: *Caputo fractional derivative*

$$D^\alpha f(x)=\frac{J^{\phi-\alpha}\left(d^\phi f(x)\right)}{dx^\phi},\ \phi-1<\alpha<\phi,\ \phi\in\mathbb{N} \tag{7}$$

In (6), Riemann-Liouville compute first, the fractional integral of the function and thereafter, an ordinary derivative of the obtained result but the reverse is the case in Caputo sense of fractional derivatives; this allows the inclusion of the traditional initial and boundary conditions in the formulation of the problem. The link between the Riemann-Liouville operator and the Caputo fractional differential operator [17, 19] is:

$$\left(J^\alpha D_t^\alpha\right)h(t)=\left(D_t^{-\alpha}D_t^\alpha\right)h(t)=h(t)-\sum_{k=0}^{n-1}h^k(0)\frac{t^k}{k!} \tag{8}$$

$$n-1<\alpha<n,\ n\in\mathbb{N}$$

As such,

$$h(t)=\left(J^\alpha D_t^\alpha\right)h(t)+\sum_{k=0}^{n-1}h^k(0)\frac{t^k}{k!} \tag{9}$$

Definition 5: *The Mittag-Leffler Function*

The Mittag-Leffler function, $E_\alpha(z)$ valid in the whole complex plane is defined and denoted by the series representation as:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)},\quad \alpha\geq 0,\ z\in\mathbb{C}\ , \tag{10}$$

$$E_{\alpha=1}(z)=e^z \quad \text{for } \alpha=1,$$

III. The Overview of the Modified Differential Transform Method (MDTM)

The differential transformation method (DTM) has been studied by many researchers and showed to be easier in terms of application when solving both linear and nonlinear differential equations as it converts the said problems to their equivalents in algebraic recursive forms [6, 7, 9, 11, 15, 20]. This is unlike other semi-analytical methods: ADM, VIM, HAM and so on that require the determination of a successive term only by integrating a previous component.

In spite of the copious merits of the DTM over other semi-analytical methods, some levels of difficulties are still encountered when dealing mainly with nonlinearity of differential equations. This again creates rooms for modification of the DTM in various forms by many authors and researchers [21,22].

Let $\psi(x,t)$ be an analytic function at (x_*,t_*) in a domain D, then in considering the Taylor series expansion of $\psi(x,t)$, regard is given to some variables $s^{ov}=t$ instead of all the variables as in the classical DTM. Thus, the MDTM of $\psi(x,t)$ with respect to t at t_* is defined and denoted by:

$$\Psi(x,h)=\frac{1}{h!}\left[\frac{\partial^h\psi(x,t)}{\partial t^h}\right]_{t=t_*} \tag{11}$$

And as such:

$$\psi(x,h)=\sum_{h=0}^{\infty}\Psi(x,h)(t-t_*)^h \tag{12}$$

Equation (12) is called the modified differential inverse transform of $\Psi(x,h)$ with respect to t .

A. Basic Theorems and properties of the MDTM [21].

Theorem a: If $\psi(x,t)=\alpha\psi_a(x,t)\pm\beta\psi_b(x,t)$, then

$\Psi(x,h)=\alpha\Psi_a(x,h)\pm\beta\Psi_b(x,h)$

Theorem b: If $\psi(x,t)=\dfrac{\alpha\partial^n\psi_*(x,\mathrm{t})}{\partial t^n}$, then

$\Psi(x,h)=\dfrac{\alpha(h+n)!}{h!}\Psi_*(x,h+n)$

Theorem c: If $\psi(x,t)=\dfrac{p(x)\partial^n\psi_*(x,\mathrm{t})}{\partial x^n}$, then

$\Psi(x,h)=\dfrac{p(x)\partial^n\Psi_*(x,h)}{\partial x^n}$

Theorem d: (MDTM of a fractional derivative)

If $g(x,t)=D_t^\alpha\varphi(x,t)$, then

$$\Gamma\left(1+\frac{k}{q}\right)G(x,k)=\Gamma\left(1+\alpha+\frac{k}{q}\right)\Phi(x,k+\alpha q),$$

and:

$$\Gamma\left(1+\alpha+\frac{k}{q}\right)\Phi(x,k+\alpha q)=\Gamma\left(1+\frac{k}{q}\right)G(x,k) \tag{13}$$

Setting $\alpha q=1$ in (13) yields (14) and (15) as follows:

$$\Phi(x,k+1)=\frac{\Gamma(1+\alpha k)}{\Gamma(1+\alpha(1+k))}G(x,k) \tag{14}$$

As such, for $\varphi(x,t)$, α-analytic at $x_0=0$

$$\varphi(x,t)=\sum_{h=0}^{\infty}\Phi(x,h)t^{\alpha h} \tag{15}$$

B. Analysis of the Fractional MDTM

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Consider the nonlinear fractional differential equation (NLFDE):

$$D_t^{\alpha}\varphi(x,t)+L_{[x]}\varphi(x,t)+N_{[x]}\varphi(x,t)-q(x,t)=0 \qquad (16)$$
$$w(x,0)=g(x),\ t>0$$

where $D_t^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Caputo derivative of $\varphi=\varphi(x,t)$; whose modified differential transform is $\Phi(x,h)$, $L_{[\cdot]}$ and $N_{[\cdot]}$ are linear and nonlinear differential operators with respect to x respectively, while $q=q(x.t)$ is the source term.

We rewrite (16) as:

$$D_t^{\alpha}\varphi(x,t)=-L_{[x]}\varphi(x,t)-N_{[x]}\varphi(x,t)+q(x,t), \qquad (17)$$
$$n-1<\alpha<n\ ,\ n\in\mathbb{N}$$

Applying the inverse fractional Caputo derivative, $D_t^{-\alpha}$ to both sides of (23) and with regard to (8) gives:

$$\varphi(x,t)=g(x)+D_t^{-\alpha}\Big[-L_{[x]}\varphi(x,t)-N_{[x]}\varphi(x,t)+q(x,t)\Big],\ \varphi(x,0)=g(x) \qquad (18)$$

Thus, expanding the analytical and continuous function, $\varphi(x,t)$ in terms of fractional power series, the inverse modified differential transform of $\Phi(x,h)$ is given as follows:

$$\varphi(x,t)=\sum_{h=0}^{\infty}\Phi(x,h)t^{\alpha h}=\varphi(x,0)+\sum_{h=1}^{\infty}\Phi(x,h)t^{\alpha h}, \qquad (19)$$
$$\varphi(x,0)=g(x)$$

IV. *Illustrative Examples and Applications*

In this subsection, we will consider via the proposed method, the following initial boundary value problems (IBVPs) describing heat and heat-like PDEs of time-fractional orders.

A. ***Problem 4.1:*** *Consider the time-fractional heat and heat-like equation {[6, 10, 13] for $\alpha=1$ }:*

$$2m_t^{\alpha}=x^2m_{xx}(x,t),\ x\in(0,1),\ t\in(0,\infty) \qquad (20)$$

subject to the boundary conditions (20a) and the initial condition (20b) below:

$$m(0,t)=0,\ m(1,t)=e^t \qquad (20a)$$

$$m(x,0)=x^2 \qquad (20b)$$

Solution to problem 4.1:

We take the modified differential transform (MDT) of (20) and (20b) as follows:

$$MDT\left[2m_t^{\alpha}=x^2m_{xx}\right]\ \&\ MDT\left[m(x,0)=x^2\right] \qquad (21)$$

$$\Rightarrow$$

$$\frac{2\Gamma(1+\alpha(1+k))}{\Gamma(1+\alpha k)}M_{x,k+1}=x^2M''_{x,k},\ k\geq 0\ \&\ M_{x,0}=x^2 \qquad (22)$$

Thus,

$$M_{x,k+1}=\frac{\Gamma(1+\alpha k)x^2}{2\Gamma(1+\alpha(1+k))}M''_{x,k}, \qquad (23)$$

When $k=0$,

$$M_{x,1}=\frac{\Gamma(1)x^2}{2\Gamma(1+\alpha)}M''_{x,0}\ ,\ M''_{x,0}=2$$

$$\therefore\ M_{x,1}=\frac{x^2}{\Gamma(1+\alpha)},\ \&\ M''_{x,1}=\frac{2}{\Gamma(1+\alpha)} \qquad (24)$$

When $k=1$,

$$M_{x,2}=\frac{\Gamma(1+\alpha)x^2}{2\Gamma(1+2\alpha)}M''_{x,1}$$

$$\Rightarrow M_{x,2}=\frac{x^2}{\Gamma(1+2\alpha)}\ \&\ M''_{x,2}=\frac{2}{\Gamma(1+2\alpha)} \qquad (25)$$

When $k=2$,

$$M_{x,3}=\frac{\Gamma(1+2\alpha)x^2}{2\Gamma(1+3\alpha)}M''_{x,2}$$

$$\Rightarrow M_{x,3}=\frac{x^2}{\Gamma(1+3\alpha)},\ \&\ M''_{x,3}=\frac{2}{\Gamma(1+3\alpha)} \qquad (26)$$

It follows thus, for $k=n$, we have:

$$M_{x,n+1}=\frac{x^2}{\Gamma(1+(n+1)\alpha)},\ \&\ M''_{x,n+1}=\frac{2}{\Gamma(1+(n+1)\alpha)} \qquad (27)$$

For $\eta=n+1$ (27) becomes:

$$M_{x,\eta}=\frac{x^2}{\Gamma(1+\eta\alpha)},\ \&\ M''_{x,\eta}=\frac{2}{\Gamma(1+\eta\alpha)} \qquad (28)$$

$$\begin{aligned}\therefore\ m_{x,t}&=\sum_{h=0}^{\infty}M_{x,h}t^{\alpha h}\\&=M_{x,0}+M_{x,1}t^{\alpha}+M_{x,2}t^{2\alpha}+M_{x,3}t^{3\alpha}+\cdots\\&=x^2+\frac{x^2t^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^2t^{3\alpha}}{\Gamma(1+3\alpha)}\\&\quad+\frac{x^2t^{4\alpha}}{\Gamma(1+4\alpha)}+\cdots+\frac{x^2t^{\eta\alpha}}{\Gamma(1+\eta\alpha)}+\cdots\\&=x^2\left\{1+\sum_{\eta=1}^{\infty}\frac{(t^{\alpha})^{\eta}}{\Gamma(1+\eta\alpha)}\right\}\\&=x^2\left\{\sum_{\eta=0}^{\infty}\frac{(t^{\alpha})^{\eta}}{\Gamma(1+\eta\alpha)}\right\}\end{aligned} \qquad (29)$$

Thus, by using definition 5, (29) becomes:

$$m_{x,t}=x^2E_{\alpha}(t^{\alpha}) \qquad (30)$$

Remark: when $\alpha=1$, the exact solution is therefore:

$$m_{x,t}=x^2e^t \qquad (31)$$

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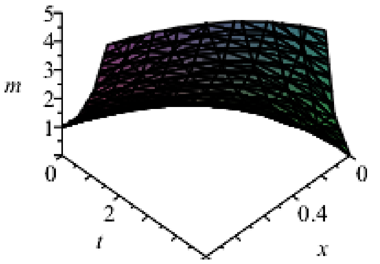

Fig. 1: Graph of the exact solution

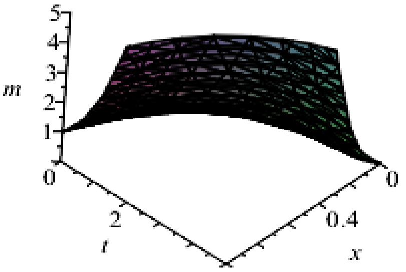

Fig. 2: Graph of the HAT solution

Fig. 1 and Fig. 2 are for *problem 4.1*. For computation, we use: $x \in (0,1)$, $t \in (0,5)$, & $m \in (0,5)$.

While Fig. 1 shows the graph of the exact solution, Fig. 2 shows the graph of the 4-term iterate solution of the HAT. This shows that the solution of Ex 3.1 in [10] is a particular case of our result.

B. ***Problem 4.2:*** *Consider the time-fractional heat and heat-like Equation {[6, 10, 13] for $\alpha = 1$}:*

$$2m_t^\alpha = y^2 m_{xx} + x^2 m_{yy},\ x, y \in (0,1),\ t \in (0,\infty) \tag{32}$$

subject to the Neumann boundary conditions (32a) and the initial condition (32b) below:

$$\begin{aligned} &m_x(0,y,t) = 0,\ m_x(1,y,t) = 2\sinh t \\ &m_y(x,0,t) = 0,\ m_y(x,1,t) = 2\cosh t \end{aligned} \tag{32a}$$

$$m(x,y,0) = y^2 \tag{33}$$

Solution procedure to problem 4.2:

We take the modified differential transform (MDT) of (32) and (33) as follows:

$$\begin{aligned} &MDT\left[m_t^\alpha = \frac{y^2}{2} m_{xx} + \frac{x^2}{2} m_{yy} \right] \ \& \\ &MDT\left[m(x,y,0) = y^2 \right] \end{aligned} \tag{34}$$

$\Rightarrow$

$$M_{x,y,k+1} = \frac{\Gamma(1+\alpha k)}{\Gamma(1+\alpha(1+k))} \left\{ \frac{y^2}{2} M''_{x,0,k} + \frac{x^2}{2} M''_{0,y,k} \right\} \tag{35}$$

such that: $\quad M''_{x,0,k} = M''_{x,k} \quad (36)$

Subsequently, we will take $M''_{x,0,k} = M''_{x,k}$ & $M''_{0,y,k} = M''_{y,k}$.

Hence, from (36):

$$M''_{x,0} = 0 \ \& \ M''_{y,0} = 2. \tag{37}$$

When $k = 0$,

$$M_{x,y,1} = \frac{\Gamma(1)}{\Gamma(1+\alpha)} \left\{ \frac{y^2}{2} M''_{x,0} + \frac{x^2}{2} M''_{y,0} \right\}$$

$\Rightarrow$

$$M_{x,y,1} = \frac{x^2}{\Gamma(1+\alpha)},\ M''_{x,1} = \frac{2}{\Gamma(1+\alpha)}, \& \ M''_{y,1} = 0 \tag{38}$$

When $k = 1$,

$$M_{x,y,2} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left\{ \frac{y^2}{2} M''_{x,1} + \frac{x^2}{2} M''_{y,1} \right\}$$

$\Rightarrow$

$$M_{x,y,2} = \frac{y^2}{\Gamma(1+2\alpha)}, M''_{y,2} = \frac{2}{\Gamma(1+2\alpha)},\ M''_{x,2} = 0 \tag{39}$$

When $k = 2$,

$$M_{x,y,3} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left\{ \frac{y^2}{2} M''_{x,2} + \frac{x^2}{2} M''_{y,2} \right\}$$

$$\Rightarrow \begin{aligned} &M_{x,y,3} = \frac{x^2}{\Gamma(1+3\alpha)}, \\ &M''_{x,3} = \frac{2}{\Gamma(1+3\alpha)},\ M''_{y,3} = 0 \end{aligned} \tag{40}$$

When $k = 3$,

$$M_{x,y,4} = \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left\{ \frac{y^2}{2} M''_{x,3} + \frac{x^2}{2} M''_{y,3} \right\}$$

$$\Rightarrow \begin{aligned} &M_{x,y,4} = \frac{y^2}{\Gamma(1+4\alpha)}, \\ &M''_{x,4} = 0,\ M''_{y,4} = \frac{2}{\Gamma(1+4\alpha)} \end{aligned} \tag{41}$$

$\therefore$

$$\begin{aligned} m_{x,y,t} &= \sum_{h=0}^{\infty} M_{x,y,h} t^{\alpha h} \\ &= M_{x,y,0} + M_{x,y,1} t^\alpha + M_{x,y,2} t^{2\alpha} \\ &\quad + M_{x,y,3} t^{3\alpha} + \cdots + M_{x,y,v} t^{v\alpha} + \cdots \\ &= y^2 + \frac{x^2 t^\alpha}{\Gamma(1+\alpha)} + \frac{y^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad + \frac{x^2 t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{y^2 t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots \end{aligned}$$

$$\begin{aligned} &= y^2 \left(1 + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots \right) \\ &\quad + x^2 \left(\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots \right) \end{aligned} \tag{42}$$

$\therefore$

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$$m_{x,y,t} = y^2 \sum_{\eta=0}^{\infty} \frac{t^{2\eta\alpha}}{\Gamma(1+2\eta\alpha)} + x^2 \sum_{\eta=0}^{\infty} \frac{t^{(2\eta+1)\alpha}}{\Gamma(1+(2\eta+1)\alpha)} \quad (43)$$

Remark: when $\alpha = 1$, (43) yields the exact solution to problem 4.2 as:

$$m_{x,y,t} = x^2 \sinh t + y^2 \cosh t \quad (44)$$

This is in agreement with {[6, 10, 13] for $\alpha = 1$}.

V. Concluding Remarks

In this paper, we implemented a handy approximation technique as a modified DTM (MDTM) for the solutions of time-fractional heat and heat-like equations. For the efficiency and reliability of the proposed technique, some illustrative examples were used; both closed-form and approximate solutions were obtained. The solutions were very much in agreement. A simple recursive equation was obtained via the proposed technique. We therefore, conclude that MDTM boosts the effectiveness of the computational work when compared with the classical DTM, even without given up accuracy. Consequently, we recommend the technique for solving linear and nonlinear time-space-fractional PDEs with applications in other areas of pure and applied sciences, finance and management.

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AUTHOR CONTRIBUTIONS

The concerned authors: SOE, GOA and AOA contributed positively to this work. They all read and approved the final manuscript for publication.

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