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Additive Manufacturing of Conical Viscoelastic Parts under Axial Tension–Compression

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***Abstract*—Piecewise-continuous additive manufacturing (AM) processes of sufficiently long in the axial direction parts of a conical shape under action of end loads are studied. The loads are statically equivalent to the axial tension–compression by some time-varying force. The being formed parts exhibit properties of deformation heredity and aging. On the basis of the approaches of mechanics of growing solids a nonclassical boundary value problem of the linear theory of viscoelasticity of the homogeneously aging isotropic media to describe the modelled process with the integral satisfaction of the force condition on the end surface of the formed solid is stated. A lemma about the possibility to carry in terms of the work objectives the product of the operator of differentiation with respect to time and the integral operator of viscoelasticity with a limit of time integration depending on solid point through the sign of integral over an arbitrary, expanding due to the growth, surface inside or on the boundary of the growing solid is proved. With its help a closed analytical solution of the stated problem of growing solids mechanics is built. This solution allows to retrace the evolution of the stress-strain state of the solid under consideration during and after the process of its additive formation.**

***Index Terms*—additive forming, conic shape, growing solid, tension–compression, viscoelasticity.**

I. Introduction

THE additive formation of solids is realized in a wide variety of natural and technological processes. Many of these processes should be considered as continuous growing processes, such that during the formation of a solid an infinitely thin layer of additional material joins to its surface each infinitely small period of time. In the course of additive processes different factors influence on solids being formed and cause their deformation. The development of stress-strain state of such solids is impossible to describe within the framework of classical concepts of continuum mechanics in principle. This is due to the lack of any configuration of the continuously growing solid which could be associated with introduction of the strain measures. An adequate description of mechanical behavior of solids deforming in processes of their continuous growing can be given on the basis of approaches and methods of mechanics of growing solids being actively developed nowadays [1], [2]. Statements and solutions of various problems on growing solids deformation can be found, for example, in [3]–[12].

The present work is devoted to the studying of additive manufacturing processes for the relatively long in the axial direction conical partss. It is assumed that in the process of formation of the part its end surfaces are acted upon by loads which are statically equivalent to the time dependant axial tension–compression. Forming the solid under consideration is carried out by means of its thickening in the radial direction due to the influx of additional material to the conic side surface. This process is piecewise-continuous, i.e. consists of arbitrary number of stages of continuous accretion alternating with arbitrary long pauses during which the influx of the material does not take place.

In the proposed study we consider the situation when the solid being formed exhibits the properties of deformation heredity (viscoelasticity) and aging (weakening the deformation properties over time regardless stresses existing in the solid), and therefore, during pauses in the growing process as well as after the final cessation of growth the solid continues to change its stress-strain state. This situation is quite difficult to simulate as rheological manifestations in the deformation response of the material continuously interact with mechanical reactions of the solid on the developing in time process of adding new material elements to it [13]–[15].

The problem is solved in quasistatic statement in the approximation of small strains. The latter let us consider the radii of the growing solid ends expanding due to the influx of additional material to be known functions of time, which are prescribed by a specific simulated growing process. The process itself is considered to be those that the additional material influx to the surface of the formed solid does not acquire nonzero stresses near this surface at the time moment of the material inclusion in the composition of this solid. The difference between the radius of one of the solid end and the radius of the other one may change arbitrarily during the process of the body growing both in size and in sign.

II. Constitutive Relations

We will consider uniform isotropic linearly viscoelastic aging material described by the equation of state [4], [16]

$$\mathbf{T}(\mathbf{r},t) = \mathcal{H}^{-1}_{\tau_0(\mathbf{r})}\big[2\mathbf{E}(\mathbf{r},t) + (\varkappa-1)\mathbf{1}\operatorname{tr}\mathbf{E}(\mathbf{r},t)\big]. \qquad (1)$$

Here $\tau_0(\mathbf{r})$ is the time when stresses appear at some point of a solid with the position vector $\mathbf{r}$; $\mathbf{T}$ and $\mathbf{E}$ are the stress and linear strain tensors, $\mathbf{1}$ is the unit tensor of the second rank; $\varkappa = (1-2\nu)^{-1}$, where $\nu = \mathrm{const}$ is Poisson's ratio. The linear operator $\mathcal{H}^{-1}_s = G(t)\big(\mathcal{I}+\mathcal{N}_s\big)$ is inverse to the linear

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operator $\mathcal{H}_s = \big(\mathcal{I} - \mathcal{L}_s\big)\, G(t)^{-1}$ with the real parameter $s \geqslant 0$, where $G(t)$ is the elastic shear modulus, $\mathcal{I}$ is the identity operator,

$$\begin{Bmatrix} \mathcal{L}_s \\ \mathcal{N}_s \end{Bmatrix} f(t) = \int_s^t f(\tau) \begin{Bmatrix} K \\ R \end{Bmatrix}(t,\tau)\, d\tau,$$

$$K(t,\tau) = G(\tau)\, \frac{\partial \Delta(t,\tau)}{\partial \tau}, \quad \Delta(t,\tau) = \frac{1}{G(\tau)} + \omega(t,\tau).$$

$K(t,\tau)$ and $R(t,\tau)$ are the kernels of creep and relaxation, $\Delta(t,\tau)$ and $\omega(t,\tau)$ are the specific strain function and the creep measure for pure shear ($t \geqslant \tau \geqslant 0$). It is accepted by definition $\omega(\tau,\tau) \equiv 0$. Taking this into account we have the identity $\mathcal{H}_\tau^{-1} \Delta(t,\tau) \equiv 1$.

In our case the stated equation (1) is used to describe the mechanical behavior of a solid which is built up by

additional material. Obviously, in this case, the function $\tau_0(\mathbf{r})$ in (1) will be determined in the following way. In the originally existing (before accreting) part of the solid it will be identically equal to the time moment t_0 of loading of this part. In the additional part of the solid, formed during accreting, it will coincide with the distribution $\tau_*(\mathbf{r})$ of moments of attaching particles $\mathbf{r}$ of additional material to the solid.

Hereinafter we will use the following notation. For arbitrary functions $g(\mathbf{r},t)$ of solid point $\mathbf{r}$ and time t and for arbitrary function of time $f(t)$ which is not associated with specific points of considered solid, we denote:

$$g^\circ(\mathbf{r},t) = \mathcal{H}_{\tau_0(\mathbf{r})}\, g(\mathbf{r},t), \quad f^\circ(t) = \mathcal{H}_{t_0} f(t). \tag{2}$$

It is necessary to note that the defining relations above were developed especially for the description of processes of concrete deformation. However, they are also well suited to describe the mechanical behaviour of some rocks, as well as polymers, soils, ice.

III. Statement of the Problem

Let there be a conical solid of rotation which length l significantly exceeds its transverse dimensions. It is made from isotropic homogeneous aging linearly viscoelastic material subordinated to the constitutive equation (1). Take the moment of this material nucleation be the start of timing t.

At the moment $t = t_0$ a load is applied to the ends of the existing solid. We believe that at every moment of time $t \geqslant t_0$ it is statically equivalent to axial forces acting in the central points of the ends and varying with time following the law $P(t)$. We will consider positive the magnitude of tensile end force.

Some time after the application loading at the time $t = t_1$ we start the process of gradual axisymmetric thickening of the considered conical solid by adding the additional material to its lateral initially free from stresses surface. Thickening occurs in such a way that in each time moment the accreted body maintains the shape of a right circular truncated cone of length l. This process is piecewise continuous in time, i.e. it consists of N consecutive phases of continuous accreting $t \in [t_{2k-1}, t_{2k})$ ($k = \overline{1,N}$), separated by pauses of arbitrary duration. At the stages of continuous accreting an infinitely thin layer of material attaches to the solid each infinitely small period of time. The added material is supposed identical to the original one. In pauses the influx of additional material to the solid does not take place and its lateral surface is free from stresses. In the process of piecewise continuous accreting and after its completion time-varying central axial forces $P(t)$ continue to act to the end surfaces of the cone.

Let us investigate the evolution of stress-strain state of the considered conical solid under specified conditions of loading before the start, during and upon the completion of the described process of accreting. The process of deformation is assumed quasi-static, and strains developing — small.

Changing the geometry of the considered conical solid due to its piecewise-continuous accreting is completely defined obviously by defining laws of increasing the radii of its ends in time. Denote them by $a(t)$ and $b(t)$, $t \geqslant t_0$. These functions are continuous, non-decreasing and constant outside intervals $[t_{2k-1}, t_{2k})$.

Combine the reference plane of a cylindrical polar coordinate system with that end of the cone which radius was denoted by $a(t)$. Place the beginning of coordinates O in the center of this end and extend coordinate axis Oz perpendicular to it inside the cone. Denote the polar radius and the angle as ρ and φ. If $\{\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{k}\}$ is normalized local basis of the introduced cylindrical coordinate system (ρ, φ, z), then the radius-vector of an arbitrary point of the solid can be represented in the form $\mathbf{r} = \mathbf{e}_\rho(\varphi)\, \rho + \mathbf{k}\, z$.

Moving due to the influx of additional material (accreting) the lateral surface of the cone under consideration is described by the equation $\rho = \Lambda(z,t)$, where $\Lambda(z,t) = a(t) \cdot (1 - z/l) + b(t) \cdot z/l$. The trace of its passing in the space forms an additional part of the considered solid. At time moments $t \in [t_{2k-1}, t_{2k})$ ($k = \overline{1,N}$) the lateral surface represents the actual growing surface of the cone, i.e. it is the level surface t of the function $\tau_*(\mathbf{r})$. Unit vectors of the external (directed from the axis of the cone) normal line to this surface form a vector field $\mathbf{n}(\mathbf{r}) = \mathbf{e}_\rho(\varphi) \cos\alpha\big(\tau_*(\mathbf{r})\big) - \mathbf{k} \sin\alpha\big(\tau_*(\mathbf{r})\big)$, in the additional part of the solid, where $\alpha(t) = \arctan\{[b(t) - a(t)]/l\}$ is the current polarstar angle of the growing cone.

IV. Boundary Value Problem for the Stage Before the Start of AM

Before the start of AM the stress-strain state of the considered conical part can be determined on the basis of the theory of viscoelasticity of homogeneously aging isotropic solids [4] and the principle of Saint-Venant from the solution of the following classical mechanical boundary value problem with integral force condition on its end surface, $t_0 \leqslant t \leqslant t_1$:

$$\begin{gathered} \nabla \cdot \mathbf{T} = \mathbf{0}, \quad 0 \leqslant \rho < \Lambda(z,t_0), \quad 0 \leqslant \varphi < 2\pi; \\ \mathcal{H}_{t_0} \mathbf{T} = 2\mathbf{E} + (\varkappa - 1)\mathbf{1}\operatorname{tr}\mathbf{E}, \quad \mathbf{E} = (\nabla \mathbf{u}^{\mathrm{T}} + \nabla \mathbf{u})/2; \\ \mathbf{n} \cdot \mathbf{T} = \mathbf{0}, \quad \rho = \Lambda(z,t_0); \\ \int_{\{z=l\}} \left\| \begin{matrix} \mathbf{k} \cdot \mathbf{T} \\ \mathbf{e}_\rho\, \rho \times (\mathbf{k} \cdot \mathbf{T}) \end{matrix} \right\| dS = \left\| \begin{matrix} \mathbf{k}\, P(t) \\ \mathbf{0} \end{matrix} \right\|; \\ \mathbf{u} = \mathbf{0}, \quad \nabla \times \mathbf{u} = \mathbf{0}, \quad \rho = 0, \quad z = 0. \end{gathered} \tag{3}$$

Here $\mathbf{u}(\mathbf{r},t)$ is the vector field of displacements. To exclude displacement components not causing deformation of the solid we imposed conditions of fixing the neighborhood of the center point of one of its end surfaces. We require these conditions to be satisfied after the start of the process of the considered solid accretion as well.

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Using the notation (2) the boundary value problem (3) can be reformulated for values $\mathbf{u}$, $\mathbf{E}$, $\mathbf{T}^\circ$, $t_0 \leqslant t \leqslant t_1$:

$$\begin{gathered}\nabla\cdot\mathbf{T}^\circ=\mathbf{0},\quad 0\leqslant\rho<\Lambda(z,t_0),\quad 0\leqslant\varphi<2\pi;\\ \mathbf{T}^\circ=2\mathbf{E}+(\varkappa-1)\mathbf{1}\operatorname{tr}\mathbf{E},\quad \mathbf{E}=(\nabla\mathbf{u}^{\mathrm{T}}+\nabla\mathbf{u})/2;\\ \mathbf{n}\cdot\mathbf{T}^\circ=\mathbf{0},\quad \rho=\Lambda(z,t_0);\\ \int_{\{z=l\}}\left\|\begin{matrix}\mathbf{k}\cdot\mathbf{T}^\circ\\ \mathbf{e}_\rho\,\rho\times(\mathbf{k}\cdot\mathbf{T}^\circ)\end{matrix}\right\|dS=\left\|\begin{matrix}\mathbf{k}\,P(t)\\ \mathbf{0}\end{matrix}\right\|;\\ \mathbf{u}=\mathbf{0},\quad \nabla\times\mathbf{u}=\mathbf{0},\quad \rho=0,\quad z=0.\end{gathered}\tag{4}$$

In the boundary value problem (4) time t is not a significant variable but acts only as a parameter.

V. Boundary Value Problem for the Stage of Piecewise-Continuous AM

A. Reduction to the Rate Characteristics of the Deformation Process

Due to the objective lack of natural (unstressed) configuration in the growing solid the kinematic description of the process of its deformation that is traditional in the mechanics of deformable solids is not suitable for this solid. However, it is clear that the particles of the new material after the attaching to the surface of growth continue to move as a part of continuous, even though growing, solid. This means that in the region of space occupied by the whole growing solid at this time, the enough smooth velocity field $\mathbf{v}(\mathbf{r},t)$ of the motion of its particles is uniquely determined. Therefore, the problem of such a body deformation can be put in terms of velocity. In this case in the formulation of the defining relations of the material a tensor of velocities of deformation $\mathbf{D}(\mathbf{r},t)=(\nabla\mathbf{v}^{\mathrm{T}}+\nabla\mathbf{v})/2$ may play a part of the deforming process characteristics. The adopted equation of state (1) can be rewritten by using this tensor in the form [13]:

$$\mathbf{S}=2\mathbf{D}+(\varkappa-1)\mathbf{1}\operatorname{tr}\mathbf{D},\tag{5}$$

where we have introduced the so-called tensor of velocities of operator stresses $\mathbf{S}(\mathbf{r},t)=\partial\mathbf{T}^\circ/\partial t$.

The approach requires knowing the whole story of changing the state of additional material elements up to their inclusion in the composition of the solid considered. In the studied in the present work process of accreting the additional material is supposed to be initially free of stresses (see Section III). In other words, we believe that the additional material begins to deform directly in the time of its attaching to the formed body, and the attaching layers of additional material to the surface of the body does not cause the appearance nonzero stresses in the formed solid near the surface of its growth:

$$\mathbf{T}=\mathbf{0},\quad \rho=\Lambda(z,t),\quad t\in[t_{2k-1},t_{2k})\ \ (k=\overline{1,N}).\tag{6}$$

Note that condition (6) provides the equality to zero of the stress vector $\mathbf{n}\cdot\mathbf{T}$ at the current growth surface, i.e. unload of this surface.

The condition of instantaneous local equilibrium in the growing body has obviously the same form as in the classical solid of permanent composition. In the considered case of mass forces absence this condition is expressed by the standard equation

$$\nabla\cdot\mathbf{T}=\mathbf{0}.\tag{7}$$

It is possible to show [13] that for the simulated growth process (in the absence of load on the future and the actual surface of solid growth during the whole process of its deformation) this equation generates similar differential equations for the tensors $\mathbf{T}^\circ$ $\mathbf{S}$:

$$\nabla\cdot\mathbf{T}^\circ=\mathbf{0},\quad \nabla\cdot\mathbf{S}=\mathbf{0}.\tag{8}$$

Equations (8)are fair at every moment of time $t>t_1$ in the region of space occupied by the whole growing body at this moment. It should be emphasized that these equations are not a trivial consequence of the equilibrium equation (7), as in the case of growing the body the integral operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and the operator of divergence $(\nabla\cdot)$ do not commute in general because of the principal dependence of time τ_0 of the occurrence of stresses in the growing solid from the point of this solid $\mathbf{r}$.

One can also show, following [13] that from the specific boundary condition (6) on the moving surface of growth $\rho=\Lambda(z,t)$ the condition on the components of the tensor $\mathbf{S}$ implies for every kth step of continuous accreting which is similar in appearance to the standard boundary condition for the stresses:

$$\mathbf{n}\cdot\mathbf{S}=\mathbf{0},\quad \rho=\Lambda(z,t),\quad t\in[t_{2k-1},t_{2k}).\tag{9}$$

Indeed, the set of conditions (6) on the time interval $t\in[t_{2k-1},t_{2k})$ can be written in the form of the initial condition in that part of the solid, which is formed on the k-th stage of its continuous growth:

$$\begin{gathered}\mathbf{T}(\mathbf{r},t)=\mathbf{0},\quad t=\tau_*(\mathbf{r}),\\ \Lambda(z,t_{2k-1})\leqslant\rho<\Lambda(z,t_{2k}).\end{gathered}\tag{10}$$

According to the definition of the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ the condition (10) is equivalent to identity

$$\mathbf{T}^\circ\big(\mathbf{r},\tau_*(\mathbf{r})\big)\equiv\mathbf{0}\tag{11}$$

in the specified part of the solid. Acting on the identity (11) with the operator of divergence we get

$$\mathbf{0}\equiv\big[\nabla\cdot\mathbf{T}^\circ(\mathbf{r},t)\big]\big|_{t=\tau_*(\mathbf{r})}+\nabla\tau_*(\mathbf{r})\cdot\mathbf{S}\big(\mathbf{r},\tau_*(\mathbf{r})\big).$$

Attracting the first equation (8) and The geometric identity $\mathbf{n}=\nabla\tau_*/|\nabla\tau_*|$ (see Section III), we get the condition (9).

In the pauses between stages of continuous growth and after the completion of growing the non-traditional condition (6)on the lateral surface of the cone should be replaced by the classical condition of equality to zero of the stress vector on this surface: $\mathbf{n}\cdot\mathbf{T}=\mathbf{0}$. Acting on this condition with the operator $\mathcal{H}_{\tau_0(\mathbf{r})}$ and differentiating the result by time t, we see that the boundary condition (9) saves force even out of time intervals $[t_{2k-1},t_{2k})$. However, it has a completely different mechanical nature in this case.

B. Transformation of the Integral Force Conditions at the End Surfaces

On the end surface of the cone after the start of its piecewise continuous accreting it is necessary to use the same integral force conditions as in the problem (3) before the accretion. However, now the the region of integration depends on time t and we need to solve a separate mathematical problem to perform the needed transition from the original conditions to the conditions on the components of the tensor

$\mathbf{S}$. The solution to this problem is obtained on the basis of the following supporting statement of a general nature.

Lemma. *Let Ω_0 and $\Omega_{\rm A}$ be two arbitrary limited surfaces inside or on the boundary of a solid, subordinated to the state equation* (5) *and formed in a process of piecewise-continuous accretion in N stages $t \in [t_{2k-1}, t_{2k})$ ($k = \overline{1, N}$) of continuous growth with arbitrary long pauses between them. The surface Ω_0 is entirely within the original (existing before accreting) part of the solid considered. The surface $\Omega_{\rm A}$ lies entirely in the additional part of the solid and is obtained by motion in space of a curve $\Gamma(t)$, $t \in [t_1, +\infty)$, which belongs to the current growth surface of the solid at every moment of its continuous accreting and is fixed in the pauses between the stages of continuous accreting, i.e. outside the time intervals $[t_{2k-1}, t_{2k})$. Let $g(\mathbf{r}, t)$ be an arbitrary function defined in the points $\mathbf{r}$ of surfaces Ω_0 and $\Omega_{\rm A}$ for $t \geqslant \tau_0(\mathbf{r})$. Then, when $t > t_1$ the formula*

$$\frac{\partial}{\partial t}\left[\int_{\Omega(t)} g(\mathbf{r}, t)\, dS\right]^{\circ} = \int_{\Omega(t)} \frac{\partial g^{\circ}(\mathbf{r}, t)}{\partial t}\, dS + \int_{\Gamma(t)} \frac{g_*(\mathbf{r})\, \upsilon(\mathbf{r}, t)}{G(t)}\, ds \qquad (12)$$

will be fair, where $g_*(\mathbf{r}) = g\big(\mathbf{r}, \tau_*(\mathbf{r})\big)$ are initial values of the function g in the points of surface $\Omega_{\rm A}$; $\upsilon(\mathbf{r}, t)$ is the normal to the curve $\Gamma(t)$ component of the velocity of its motion along the surface $\Omega_{\rm A}$, calculated at the point $\mathbf{r} \in \Gamma(t)$; the expanding in time (disconnected in general) surface $\Omega(t)$ combines the surface Ω_0 and that part of the surface $\Omega_{\rm A}$, which has already been formed by the time $t \geqslant t_0$:

$$\Omega(t) = \Omega_0 \cup \begin{cases} \varnothing, & t \in [t_0, t_1], \\ \big\{\Gamma(\tau)\,\big|\, t_1 \leqslant \tau \leqslant t\big\}, & t \in (t_1, +\infty). \end{cases}$$

We omit the proof of this statement due to the limited volume of this paper. Note that the surfaces Ω_0 and $\Omega_{\rm A}$ considered in the Lemma may have arbitrary curvature. Meanwhile their boundaries may not have common points. In the cpecial case it is possible that $\Omega_0 = \varnothing$. Forming a surface $\Omega_{\rm A}$ curves $\Gamma(t)$ can be both closed and unclosed. In particular, the surface $\Omega_{\rm A}$ may "circle" original part of the solid or form a "tube" enveloping only the material of the additional part of the having been formed solid.

C. Formulation of the Boundary Value Problem

As a surface $\Omega(t)$ from the Lemma in the being solved problem of accreting a conical solid it is necessary to consider the flat surface constituting one end side of the growing cone $z = l$ for $t \geqslant t_0$. The surface $\Omega_{\rm A}$ in this case is annular, and its forming curves $\Gamma(t)$ are concentric circles $\rho = b(t)$. The surface Ω_0 is a circle $0 \leqslant \rho \leqslant b_0$. Then by the Lemma because of the condition (10) we have

$$\frac{\partial}{\partial t}\left[\int_{\{z=l\}} \left\|\begin{matrix} \mathbf{k}\cdot\mathbf{T} \\ \mathbf{e}_\rho\, \rho \times (\mathbf{k}\cdot\mathbf{T}) \end{matrix}\right\| dS\right]^{\circ} = \int_{\{z=l\}} \left\|\begin{matrix} \mathbf{k}\cdot\mathbf{S} \\ \mathbf{e}_\rho\, \rho \times (\mathbf{k}\cdot\mathbf{S}) \end{matrix}\right\| dS, \quad t > t_1.$$

Thus, collecting together all the above-formulated relations for the quantities $\mathbf{v}$, $\mathbf{D}$, $\mathbf{S}$ we can supply the following boundary value problem describing the process of deforming the considered conical solid on all the temporary beam after the beginning of its accreting, $t > t_1$:

$$\begin{gathered} \nabla\cdot\mathbf{S} = \mathbf{0}, \quad 0 \leqslant \rho < \Lambda(z, t), \quad 0 \leqslant \varphi < 2\pi; \\ \mathbf{S} = 2\mathbf{D} + (\varkappa - 1)\mathbf{1}\,\mathrm{tr}\,\mathbf{D}, \quad \mathbf{D} = (\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})/2; \\ \mathbf{n}\cdot\mathbf{S} = \mathbf{0}, \quad \rho = \Lambda(z, t); \\ \int_{\{z=l\}} \left\|\begin{matrix} \mathbf{k}\cdot\mathbf{S} \\ \mathbf{e}_\rho\, \rho \times (\mathbf{k}\cdot\mathbf{S}) \end{matrix}\right\| dS = \left\|\begin{matrix} \mathbf{k}\,\partial P^{\circ}(t)/\partial t \\ \mathbf{0} \end{matrix}\right\|; \\ \mathbf{v} = \mathbf{0}, \quad \nabla\times\mathbf{v} = \mathbf{0}, \quad \rho = 0, \quad z = 0. \end{gathered} \qquad (13)$$

Given in (13) conditions for the vector field of velocities $\mathbf{v}(\mathbf{r}, t)$ in the neibourhood of the coordinates origin O provide a rigid fixing this neighbourhood throughout the whole process of deformation of considered growing solid.

VI. Solution of an Auxiliary Problem for the Tension-Compression of a Truncated Cone

As we can see, the problem (4) and the problem (13) turned out to be mathematically equivalent to the same classical mechanical problem of the equilibrium of a linearly elastic truncated circular cone of permanent composition with free lateral surface $\rho = \Lambda(z, t)$, $z \in [0, l]$, rigidly fixed in the coordinates origin and being under the action of axial forces centrally applied to its ends. The radii of the ends of the cone and the value of forces acting on it depend on a real parameter t. This formal coincidence is gained by substituting in the problems (4) and (13) the values P° and $\partial P^{\circ}/\partial t$ to the value of tensile force related to the shear modulus, the tensors $\mathbf{T}^{\circ}$ and $\mathbf{S}$ to the stress tensor related to the shear modulus, and in the problem (13) — also the tensor $\mathbf{D}$ to the small strain tensor and the vector $\mathbf{v}$ to the displacement vector as well. Let us construct the analytical solution of the described classical problem of the theory of elasticity.

Consider a non-growing elastic truncated cone of length l, to that ends of radii a and b the central tensile axial forces of magnitude P are applied. We introduce the polar cylindrical coordinate system (ρ, φ, z) in the region busy by the cone in the way we did it in Section III for accreted conical solid. A cone is considered sufficiently long in the axial direction compared with its transverse dimensions. In this case, the specific distribution of acting on the ends forces does not influence the stress-strain state of the greater part of the cone, and this condition can be determined on the basis of the Saint-Venant principle. To do it we can use the known solution of the problem of tensioning an infinitely long pointed cone with an axial force P applied to its vertex [17]. Let us introduce an additional spherical coordinate system (R, Θ, Φ) with the center at the cone vertex, where R is the length of radius-vector, Φ is the longitudinal angle counted around the axis of symmetry of the cone, Θ is the pole angle counted from the axis of symmetry inside the solid. In this coordinate system the solution mentioned has the form:

$$\begin{gathered} \mathbf{u} = \mathbf{e}_R\, u_R + \mathbf{e}_\Theta\, u_\Theta, \\ \left\|\begin{matrix} u_R \\ u_\Theta \end{matrix}\right\| = \frac{P}{4\pi G R\, Q(\cos\Theta_0)} \times \\ \left\|\begin{matrix} 2(\varkappa + 1)\cos\Theta - (1 + \cos\Theta_0) \\ \big[(1 + \cos\Theta_0)/(1 + \cos\Theta) - (\varkappa + 2)\big]\sin\Theta \end{matrix}\right\|; \end{gathered}$$

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$$\mathbf{T} = \mathbf{e}_R \mathbf{e}_R \sigma_R + \mathbf{e}_\Theta \mathbf{e}_\Theta \sigma_\Theta + \mathbf{e}_\Phi \mathbf{e}_\Phi \sigma_\Phi + (\mathbf{e}_R \mathbf{e}_\Theta + \mathbf{e}_\Theta \mathbf{e}_R) \tau_{R\Theta},$$

$$\begin{Vmatrix} \sigma_R \\ \sigma_\Theta \\ \sigma_\Phi \\ \tau_{R\Theta} \end{Vmatrix} = \frac{P}{2\pi R^2 Q(\cos\Theta_0)} \times \begin{Vmatrix} 1 + \cos\Theta_0 - (3\varkappa + 1)\cos\Theta \\ \left[1 - (1 + \cos\Theta_0)/(1 + \cos\Theta)\right]\cos\Theta \\ \cos\Theta - (1 + \cos\Theta_0)/(1 + \cos\Theta) \\ \left[1 - (1 + \cos\Theta_0)/(1 + \cos\Theta)\right]\sin\Theta \end{Vmatrix}.$$

Here $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi\}$ is the normalized local basis of the spherical coordinate system, Θ_0 is the angle of the cone polarstar, $Q(\xi) = \varkappa\xi^3 - \xi^2 + \xi - \varkappa$.

To apply the written solutions to the considered in this section classical problem of theory of elasticity it is necessary to extend the lateral surface of the considered truncated cone of length l in both sides in the axial direction so as to obtain infinitely long cone with a vertex. Denote this vertex as O'. After this it is necessary to analyze separately the cases $a < b$ and $a > b$.

In the case $a < b$ ($a > b$) the reference end $z = 0$ of the truncated cone is closer among its two ends to the vertex (further from the vertex) O' of a pointed cone. Therefore, the introduced in Section III vector $\mathbf{k}$ is codirected (oppositely directed) to the vector defining the direction $\Theta = 0$, and the vector $\mathbf{e}_\varphi$ is codirected (oppositely directed) to the vector $\mathbf{e}_\Phi$. Thus, the transition from the additionally introduced spherical coordinate system to the original cylindrical one is maintained by means of the following transformation of the local bases:

$$\begin{Vmatrix} \mathbf{e}_R & \mathbf{e}_\Theta & \mathbf{e}_\Phi \end{Vmatrix} = \begin{Vmatrix} \mathbf{e}_\rho & \mathbf{e}_\varphi & \mathbf{k} \end{Vmatrix} \cdot \begin{Vmatrix} \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & \pm 1 \\ \pm\cos\Theta & \mp\sin\Theta & 0 \end{Vmatrix}.$$

The upper signs correspond to the case $a < b$, the lower ones — to the case $a > b$. Meanwhile, we also need to put $\cos\Theta = \pm(z + d)/R$, $\sin\Theta = \rho/R$, $R = \sqrt{\rho^2 + (z + d)^2}$, $d = la/(b - a)$, $\Theta_0 = \pm\alpha$, $\alpha = \arctan\{(b - a)/l\} = \arctan\{a/d\}$. The value d is, accurate to sign, the distance from the reference end $z = 0$ to the vertex O' of a pointed cone. The value of α is, accurate to sign, the polarstar angle of the cone.

Perform specified transformations, ensuring rigid fixing of a neighborhood of the coordinates origin O by adding a proper constant to the axial displacement. We find:

$$\mathbf{u} = \mathbf{e}_\rho u_\rho + \mathbf{e}_z u_z,$$

$$\begin{Vmatrix} u_\rho \\ u_z + c \end{Vmatrix} = \frac{P}{4\pi G R(\rho, z) Q(\cos\alpha)} \times \begin{Vmatrix} \left[\frac{\pm\varkappa(z + d)}{R(\rho, z)} - \frac{1 + \cos\alpha}{1 \pm (z + d)/R(\rho, z)}\right] \frac{\rho}{R(\rho, z)} \\ \pm\left[\varkappa(z + d)^2/R^2(\rho, z) + \varkappa + 1 - \cos\alpha\right] \end{Vmatrix},$$

$$\mathbf{T} = \mathbf{e}_\rho \mathbf{e}_\rho \sigma_\rho + \mathbf{e}_\varphi \mathbf{e}_\varphi \sigma_\varphi + \mathbf{e}_z \mathbf{e}_z \sigma_z + (\mathbf{e}_\rho \mathbf{e}_z + \mathbf{e}_z \mathbf{e}_\rho) \tau_{\rho z},$$

$$\begin{Vmatrix} \sigma_\rho \\ \sigma_\varphi \\ \sigma_z \\ \tau_{\rho z} \end{Vmatrix} = \frac{P}{2\pi R^2(\rho, z) Q(\cos\alpha)} \times$$

$$\begin{Vmatrix} \frac{1 + \cos\alpha}{1 \pm (z + d)/R(\rho, z)} \mp \left[\cos\alpha + \frac{3\varkappa\rho^2}{R^2(\rho, z)}\right] \frac{z + d}{R(\rho, z)} \\ \pm\frac{z + d}{R(\rho, z)} - \frac{1 + \cos\alpha}{1 \pm (z + d)/R(\rho, z)} \\ \pm\left[\cos\alpha - 3\varkappa(z + d)^2/R^2(\rho, z)\right](z + d)/R(\rho, z) \\ \pm\left[\cos\alpha - 3\varkappa(z + d)^2/R^2(\rho, z)\right]\rho/R(\rho, z) \end{Vmatrix}.$$

Above, the following constant is used:

$$c = \frac{(2\varkappa + 1 - \cos\alpha) P \tan\alpha}{4\pi G a\, Q(\cos\alpha)}.$$

The expressions received for the displacements and stresses can be written simplier with the function $\zeta(\rho, z) = \pm(z+d)/R(\rho, z) = \cos\Theta$, if we also enter the function of the shape of the cone lateral surface $\Lambda(z) = a\cdot(1 - z/l) + b\cdot z/l$ and note that $z + d = \Lambda(z)/\tan\alpha$. Indeed, then we have $\pm R(\rho, z) = (z + d)/\cos\Theta = \Lambda(z)/[\zeta(\rho, z)\tan\alpha]$, moreover, the function $\zeta(\rho, z)$ can be calculated by the formula

$$\zeta(\rho, z) = \left[\rho^2 \tan^2\!\alpha/\Lambda^2(z) + 1\right]^{-1/2}, \tag{14}$$

as $\zeta = (\tan^2\Theta + 1)^{-1/2}$ and $\tan\Theta = \sin\Theta/\cos\Theta = \pm\rho/(z + d) = \pm\rho\tan\alpha/\Lambda(z)$.

In result

$$\begin{Vmatrix} u_\rho \\ u_z + c \end{Vmatrix} = \frac{P\,\zeta(\rho, z)\tan\alpha}{4\pi G\,\Lambda(z)\,Q(\cos\alpha)} \times \begin{Vmatrix} \left[\varkappa\zeta(\rho, z) - \frac{1 + \cos\alpha}{1 + \zeta(\rho, z)}\right] \frac{\zeta(\rho, z)\,\rho\tan\alpha}{\Lambda(z)} \\ \varkappa\zeta^2(\rho, z) + \varkappa + 1 - \cos\alpha \end{Vmatrix},$$

$$\begin{Vmatrix} \sigma_\rho \\ \sigma_\varphi \\ \sigma_z \\ \tau_{\rho z} \end{Vmatrix} = \frac{P\,\zeta^2(\rho, z)\tan^2\!\alpha}{2\pi\,\Lambda^2(z)\,Q(\cos\alpha)} \times \tag{15}$$

$$\begin{Vmatrix} \frac{1 + \cos\alpha}{1 + \zeta(\rho, z)} - \left[\cos\alpha - 3\varkappa\zeta^2(\rho, z) + 3\varkappa\right]\zeta(\rho, z) \\ \zeta(\rho, z) - \left[1 + \cos\alpha\right]/\left[1 + \zeta(\rho, z)\right] \\ \left[\cos\alpha - 3\varkappa\zeta^2(\rho, z)\right]\zeta(\rho, z) \\ \left[\cos\alpha - 3\varkappa\zeta^2(\rho, z)\right]\zeta(\rho, z)\,\rho\tan\alpha/\Lambda(z) \end{Vmatrix}.$$

Note that all stresses are proportional to the value of $P/\left[\pi\Lambda^2(z)\right]$, which is, obviously, the average normal stress acting at any cross-section $z = \mathrm{const}$ of the cone.

It is easy to make sure that the expression (15) remain in force even in the special case of cylindrical solid, which was so far excluded from our consideration. Indeed, if the parameter α tends to zero at fixed values of other geometric parameters a and l and arbitrary fixed values of the variables $\rho \in [0, a)$ and $z \in (0, l)$, then given the submission $\Lambda(z) = a + z\tan\alpha$ we have

$$u_\rho \to -\frac{P}{\pi a^2}\cdot\frac{(\varkappa - 1)\rho}{2(3\varkappa - 1)G}, \quad u_z \to \frac{P}{\pi a^2}\cdot\frac{\varkappa z}{(3\varkappa - 1)G},$$

$$\sigma_z \to \frac{P}{\pi a^2}, \quad \sigma_{\rho,\varphi}, \tau_{\rho z} \to 0.$$

Obtained with $\alpha \to 0$ the limit values of displacements and stresses correspond, obviously, to the solution of the Saint-Venant problem of the uniaxial tensile with a force P of a cylinder with the fixed end $z = 0$.

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VII. The Construction of the Solution for an AM Problem

In Section VI the solution of the classical problem of tension–compression of a non-growing elastic conical part with an arbitrary correlation of radii of its loaded end surfaces is constructed. As indicated in Section VI, after a suitable replacement of variables contained in the solution it is possible to obtain the solutions of boundary value problems (4) and (13). These solutions will contain the introduced in Section III functions $\alpha(t)$ and $\Lambda(z,t)$ as well as the function

$$\zeta(\rho,z,t) = \left[\rho^2 \tan^2\alpha(t)/\Lambda^2(z,t) + 1\right]^{-1/2}$$

introduced by analogy with (14).

As a result, in each point of $\mathbf{r}$ of the considered piecewise continuously accreted aging viscoelastic conical solid we will know the evolution of the velocity vector $\mathbf{v}$ and the tensor $\mathbf{S}$ of velocities of operator stresses on the time beam

$$t > \tau_1(\mathbf{r}) = \begin{cases} t_1, & 0 \leqslant \rho < \Lambda(z,t_0), \\ \tau_*(\mathbf{r}), & \Lambda(z,t_0) \leqslant \rho < \Lambda(z,t_{2N}), \end{cases}$$

which covers the entire history of deformation of the neighborhood of a given point in the composition of the formed solid after the beginning of the process of its accretion. And at the points of the original part of this solid we will also know the evolution of the displacement vector $\mathbf{u}$ and the operator stress tensor $\mathbf{T}^\circ$ on the time segment $t \in [t_0, t_1]$ before the beginning of the accretion process. After that, the evolution of the operator stress tensor $\mathbf{T}^\circ$ at any point $\mathbf{r}$ of the solid for all $t \geqslant \tau_1(\mathbf{r})$ can be recovered by using the integration procedure:

$$\mathbf{T}^\circ(\mathbf{r},t) = \mathbf{T}^\circ\big(\mathbf{r},\tau_1(\mathbf{r})\big) + \int_{\tau_1(\mathbf{r})}^{t} \mathbf{S}(\mathbf{r},\tau)\,d\tau.$$

Here we have $\mathbf{T}^\circ\big(\mathbf{r},\tau_1(\mathbf{r})\big) = \mathbf{0}$ in the additional part of the solid according to the initial condition (11).

When in a point $\mathbf{r}$ of the considered solid we know the complete evolution of the tensor $\mathbf{T}^\circ$, i.e., the values of this tensor since the moment $t = \tau_0(\mathbf{r})$ of occurrence of stresses in a given point, we can find the complete evolution of the stresses tensor $\mathbf{T}$ in this point by using the inverse to $\mathcal{H}_{\tau_0(\mathbf{r})}$ transformation $\mathcal{H}^{-1}_{\tau_0(\mathbf{r})}$:

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)} = \mathbf{T}^\circ(\mathbf{r},t) + \int_{\tau_0(\mathbf{r})}^{t} \mathbf{T}^\circ(\mathbf{r},\tau)\,R(t,\tau)\,d\tau.$$

When we use a particular approximation for the creep kernel $K(t,\tau)$ the expression for the respective relaxation kernel $R(t,\tau)$ may not be known in the closed form or be too bulky. Then the procedure of reconstructing the evolution of the tensor $\mathbf{T}$ by numerical treatment of the Volterra equation

$$\frac{\mathbf{T}(\mathbf{r},t)}{G(t)} - \int_{\tau_0(\mathbf{r})}^{t} \frac{\mathbf{T}(\mathbf{r},\tau)}{G(\tau)}\,K(t,\tau)\,d\tau = \mathbf{T}^\circ(\mathbf{r},t),$$

for example, by the method of quadratures [18], will be less expensive from a computational point of view and may be even more precise.

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