Classes of Ordinary Differential Equations Obtained for the Probability Functions of Logistic and Log-Logistic Distributions

Hilary I. Okagbue, *Member,* IA*ENG*, Pelumi E. Oguntunde, Abiodun A. Opanuga and Patience I. Adamu

***Abstract*— Differential calculus was used to obtain the ordinary differential equations (ODE) of the probability density function (PDF), Quantile function (QF), survival function (SF) and hazard function (HF) of the Logistic and Log-Logistic distributions. The parameters and support that define the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method can be extended to other probability distributions, functions and can serve an alternative to estimation and approximation. Computer codes and programs can be used for the implementation.**

***Index Terms*— Differential calculus, quantile function, hazard function, reversed hazard function, inverse survival function, survival function, Logistic distribution.**

I. Introduction

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-5].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of probability distributions can be expressed as ODE whose solution is the PDF. Some of which are available. They include: beta distribution [6], Lomax distribution [7], beta prime distribution [8], Laplace distribution [9] and raised cosine distribution [10].

The aim of this paper is to develop homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF) and hazard function (HF) of the Logistic and log-Logistic distribution. The cases for the inverse survival function (ISF) and reversed hazard function (RHF) were not considered because of their complexity. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the existence of the ODEs. Similar results for other distributions have been proposed, see [11-24] for details.

Logistic is a well-known continuous distribution whose cumulative distribution function is the logistic function [25]. Dubey [26] noted that the distribution is one of special cases of compound generalized extreme distribution

Several aspects of the distribution have been studied by different researchers such as: shape of the distribution [27], approximation of the distribution to the cumulative normal distribution [28], statistical tests [29] and order statistics [30].

Bayesian inference, parameter estimation, maximum likelihood estimation about the distribution has been studied extensively. The details can be found in [31-36].

Some generalizations of the distribution includes: class of bivariate Logistic distributions by [37], generalization [38], Gumbel bivariate Logistic distribution [39], logit logistic distribution [40], skew logistic distribution [41], logistic-uniform distribution [42], some generalized Logistic-X distributions [43], half and generalized half logistic distribution of type I [44] and [45].

Applications include: testing the reliability of economic plan [46] and modeling of water demand [47].

The log-logistic distribution is the probability distribution of a random variable whose logarithm has a logistic distribution. It is similar in shape to the log-normal distribution but is characterized by heavy tails.

Differential calculus was used to obtain the results.

II. Logistic Distribution

A. Probability Density Function

The probability density function of the Logistic distribution is given in three forms;

$$f(x)=\frac{\mathrm{e}^{-\left(\frac{x-\mu}{s}\right)}}{s(1+\mathrm{e}^{-\left(\frac{x-\mu}{s}\right)})^2} \qquad (1)$$

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H. I. Okagbue, P. E. Oguntunde, A. A. Opanuga and P. I. Adamu are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
pelumi.oguntunde@covenantuniversity.edu.ng
abiodun.opanuga @covenantuniversity.edu.ng
patience.adamu@covenantuniversity.edu.ng

$$f(x)=\frac{e^{\left(\frac{x-\mu}{s}\right)}}{s(1+e^{\left(\frac{x-\mu}{s}\right)})^2} \quad (2)$$

$$f(x)=\frac{1}{4s}\operatorname{sech}^2\left(\frac{x-\mu}{2s}\right) \quad (3)$$

To obtain the first order ordinary differential equation for the probability density function of the Logistic distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\frac{-\frac{1}{s}e^{-\left(\frac{x-\mu}{s}\right)}}{e^{-\left(\frac{x-\mu}{s}\right)}}+\frac{2(1+e^{-\left(\frac{x-\mu}{s}\right)})^{-3}e^{-\left(\frac{x-\mu}{s}\right)}}{s(1+e^{-\left(\frac{x-\mu}{s}\right)})^{-2}}\right\}f(x) \quad (4)$$

$$f'(x)=\left\{-\frac{1}{s}+\frac{2e^{-\left(\frac{x-\mu}{s}\right)}}{s(1+e^{-\left(\frac{x-\mu}{s}\right)})}\right\}f(x) \quad (5)$$

$$f'(x)=-\frac{1}{s}\left\{\frac{1-e^{-\left(\frac{x-\mu}{s}\right)}}{1+e^{-\left(\frac{x-\mu}{s}\right)}}\right\}f(x) \quad (6)$$

Differentiate equation (3) to obtain a first order ODE;

$$f'(x)=-\frac{1}{4s^2}\operatorname{sech}^2\left(\frac{x-\mu}{2s}\right)\tanh\left(\frac{x-\mu}{2s}\right) \quad (7)$$

$$f'(x)=-\frac{1}{s}\tanh\left(\frac{x-\mu}{2s}\right)f(x) \quad (8)$$

Squaring both sides of the equation;

$$f'^2(x)=\frac{1}{s^2}\tanh^2\left(\frac{x-\mu}{2s}\right)f^2(x) \quad (9)$$

Appling the trigonometric identity to equation (9);

$$\tanh^2\left(\frac{x-\mu}{2s}\right)=\operatorname{sech}^2\left(\frac{x-\mu}{2s}\right)-1 \quad (10)$$

Substitute equation (10) into equation (9);

$$f'^2(x)=\frac{f^2(x)}{s^2}\left(\operatorname{sech}^2\left(\frac{x-\mu}{2s}\right)-1\right) \quad (11)$$

Equation (3) can also be simplified as;

$$4sf(x)=\operatorname{sech}^2\left(\frac{x-\mu}{2s}\right) \quad (12)$$

Substitute equation (12) into equation (11);

$$f'^2(x)=\frac{f^2(x)}{s^2}\left(4sf(x)-1\right) \quad (13)$$

The first order ODE for the probability density function of the Logistic distribution is given as;

$$s^2f'^2(x)-4sf^3(x)+f^2(x)=0 \quad (14)$$

$$f(1)=\frac{e^{-\left(\frac{1-\mu}{s}\right)}}{s(1+e^{-\left(\frac{1-\mu}{s}\right)})^2}=\frac{1}{4s}\operatorname{sech}^2\left(\frac{1-\mu}{2s}\right) \quad (15)$$

A special case is considered which showed the trigonometric nature of the Logistic distribution. This is when $\mu=0$ and $s=1$ is substituted in equation (6), to obtain;

$$f'(x=x,\mu=0,s=1)=-\left\{\frac{1-e^{-x}}{1+e^{-x}}\right\}f(x=x,\mu=0,s=1) \quad (16)$$

Higher order ODEs for the PDF of the Logistic distribution can be obtained from equation (14). See [11-24] for similar results.

B. Quantile Function

The Quantile function of the Logistic distribution is given as;

$$Q(p)=\mu+s\ln\left(\frac{p}{1-p}\right) \quad (17)$$

Differentiate equation (17);

$$Q'(p)=\frac{s}{p(1-p)} \quad (18)$$

The first order ordinary differential for the Quantile function of the Logistic distribution is given as;

$$p(1-p)Q'(p)-s=0 \quad (19)$$

$$Q\left(\frac{1}{10}\right)=\mu-2.19723s \quad (20)$$

Special cases of equation (19) are considered;

Case I; When $p=0.25$

$$3Q'(p)-16s=0 \quad (21)$$

Case II; When $p=0.50$

$$Q'(p)-4s=0 \quad (22)$$

Case III; When $p=0.75$

$$3Q'(p)-16s=0 \quad (23)$$

Differentiate equation (18), to obtain the second order ODE;

$$Q''(p)=\frac{s}{p(1-p)^2}-\frac{s}{p^2(1-p)} \quad (24)$$

Two ODEs can be obtained from further simplification of equations (24);

ODE 1

Using equation (18) in (24);

$$Q''(p)=\frac{s}{p(1-p)}\left[\frac{1}{1-p}-\frac{1}{p}\right] \quad (25)$$

$$Q''(p)=Q'(p)\left[\frac{2p-1}{p(1-p)}\right] \quad (26)$$

$$p(1-p)Q''(p)-(2p-1)Q'(p)=0 \quad (27)$$

ODE 2

Using the simplified form of equation (18) given as;

$$\frac{pQ'(p)}{s}=\frac{1}{1-p} \quad (28)$$

Substitute equation (28) into equation (26);

$$Q''(p)=Q'(p)\left[\frac{pQ'(p)}{s}-\frac{1}{p}\right] \quad (29)$$

$$spQ''(p)-p^2Q'^2(p)+sQ'(p)=0 \quad (30)$$

$$Q'\left(\frac{1}{10}\right)=\frac{100s}{9} \quad (31)$$

Differentiate equation (24), to obtain the third order ODE;

$$Q'''(p)=\frac{2s}{p(1-p)^3}-\frac{2s}{p^2(1-p)^2}+\frac{2s}{p^3(1-p)} \quad (32)$$

Three ODEs can be obtained from the simplification of equations (32);

ODE 1;

Simplify equation (32) using equation (18);

$$Q'''(p)=\frac{2s}{p(1-p)}\left[\frac{1}{(1-p)^2}-\frac{1}{p(1-p)}+\frac{1}{p^2}\right] \quad (33)$$

$$Q'''(p)=2Q'(p)\left[\frac{3p^2-3p+1}{p^2(1-p)^2}\right] \quad (34)$$

$$p^2(1-p)^2Q'''(p)-2(3p^2-3p+1)Q'(p)=0 \quad (35)$$

ODE 2;

The following equations obtained from the simplifications of equations (18) or (28) is needed to obtain the ordinary differential equation.

$$\frac{Q'(p)}{s}=\frac{1}{p(1-p)} \quad (36)$$

$$\frac{p^2Q'^2(p)}{s^2}=\frac{1}{(1-p)^2} \quad (37)$$

Substitute equations (36) and (37) into equation (33);

$$Q'''(p)=2Q'(p)\left[\frac{p^2Q'^2(p)}{s^2}-\frac{Q'(p)}{s}+\frac{1}{p^2}\right] \quad (38)$$

$$s^2p^2Q'''(p)-2p^4Q'^3(p)+2sp^2Q'^2(p)-2s^2Q'(p)=0 \quad (39)$$

ODE 3;

Equation (32) can also be written as;

$$Q'''(p)=\frac{2}{(1-p)}\left[\frac{s}{p(1-p)^2}-\frac{s}{p^2(1-p)}\right]+\frac{2}{p^2}\left[\frac{s}{p(1-p)}\right] \quad (40)$$

Substitute equations (18) and (24) into equation (40);

$$Q'''(p)=\frac{2Q''(p)}{(1-p)}+\frac{2Q'(p)}{p^2} \quad (41)$$

$$(1-p)p^2Q'''(p)-2p^2Q''(p)-2(1-p)Q'(p)=0 \quad (42)$$

See [11-24] for similar results.

C. Survival Function

The Survival function of the Logistic distribution is given as;

$$S(t)=\frac{e^{-\left(\frac{t-\mu}{s}\right)}}{(1+e^{-\left(\frac{t-\mu}{s}\right)})} \quad (43)$$

Differentiate equation (43);

$$S'(t)=\left\{\frac{-\frac{1}{s}e^{-\left(\frac{t-\mu}{s}\right)}}{e^{-\left(\frac{t-\mu}{s}\right)}}+\frac{\frac{1}{s}e^{-\left(\frac{t-\mu}{s}\right)}(1+e^{-\left(\frac{t-\mu}{s}\right)})^{-2}}{(1+e^{-\left(\frac{t-\mu}{s}\right)})^{-1}}\right\}S(t) \quad (44)$$

$$S'(t)=\left\{-\frac{1}{s}+\frac{\frac{1}{s}e^{-\left(\frac{t-\mu}{s}\right)}}{(1+e^{-\left(\frac{t-\mu}{s}\right)})}\right\}S(t) \quad (45)$$

$$S'(t)=\left\{-\frac{1}{s}+\frac{S(t)}{s}\right\}S(t) \quad (46)$$

The first order ODE for the Survival function of the Logistic distribution is given as;

$$sS'(t)-S^2(t)+S(t)=0 \quad (47)$$

$$S(1)=\frac{e^{-\left(\frac{1-\mu}{s}\right)}}{(1+e^{-\left(\frac{1-\mu}{s}\right)})} \quad (48)$$

Higher order ODEs for the survival function of the Logistic distribution can be obtained from equation (47). See [11-24] for similar results.

D. Hazard Function

The Hazard function of the Logistic distribution is given as;

$$h(t)=\frac{1}{s(1+e^{-\left(\frac{t-\mu}{s}\right)})} \quad (49)$$

Differentiate equation (49);

$$h'(t)=\frac{e^{-\left(\frac{x-\mu}{s}\right)}}{s^2(1+e^{-\left(\frac{x-\mu}{s}\right)})^2} \quad (50)$$

$$h'(t)=\frac{f(t)}{s} \quad (51)$$

$$sh'(t)-f(t)=0 \quad (52)$$

$$h(1)=\frac{1}{s(1+e^{-\left(\frac{1-\mu}{s}\right)})} \quad (53)$$

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Higher order ODEs can be obtained;

$$h''(t)=\frac{f'(t)}{s} \tag{54}$$

$$sh''(t)-f'(t)=0 \tag{55}$$

$$h'''(t)=\frac{f''(t)}{s} \tag{56}$$

$$sh'''(t)-f''(t)=0 \tag{57}$$

Using equations (51), (54) and (56) on the results of the PDF of the Logistic distribution, the following ordinary differential equations can be obtained for the Hazard function.

Equation (14) becomes;

$$s^2h''^2(t)-4sh'^3(t)+h'^2(t)=0 \tag{58}$$

See [11-24] for similar results.

III. Log-Logistic Distribution

A. Probability Density Function

The probability density function of the Log-logistic distribution is given by;

$$f(x)=\frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2} \tag{59}$$

Differentiate equation (59);

$$f'(x)=\left\{\begin{array}{l}\dfrac{\frac{\beta-1}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-2}}{\left(\frac{x}{\alpha}\right)^{\beta-1}}\\[2ex] -\dfrac{\frac{2\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-3}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-2}}\end{array}\right\}f(x) \tag{60}$$

$$f'(x)=\left\{\frac{\beta-1}{x}-\frac{\frac{2\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right\}f(x) \tag{61}$$

Special cases are considered.

Case I; When $\alpha=\beta=1$, equation (61) becomes;

$$f_1'(x)=-\frac{2}{1+x}f_1(x) \tag{62}$$

$$(1+x)f_1'(x)+2f_1(x)=0 \tag{63}$$

Case I; When $\beta=1$, equation (61) becomes;

$$f_2'(x)=-\frac{2}{\alpha+x}f_2(x) \tag{64}$$

$$(1+x)f_2'(x)+2f_2(x)=0 \tag{65}$$

Differentiate equation (61);

$$f''(x)=\left\{\begin{array}{l}-\dfrac{\beta-1}{x^2}-\dfrac{2\left(\frac{\beta}{\alpha}\right)^2\left(\left(\frac{x}{\alpha}\right)^{\beta-1}\right)^2}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2}\\[2ex] -\dfrac{2\left(\frac{\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-2}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\end{array}\right\}f(x)$$

$$+\left\{\frac{\beta-1}{x}-\frac{\frac{2\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right\}f'(x) \tag{66}$$

The following equations obtained from (61) is helpful in the simplification of equation (66).

$$\frac{f'(x)}{f(x)}=\left\{\frac{\beta-1}{x}-\frac{\frac{2\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}\right\} \tag{67}$$

$$-\frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}=\frac{1}{2}\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right] \tag{68}$$

$$\frac{\left(\frac{\beta}{\alpha}\right)^2\left(\left(\frac{x}{\alpha}\right)^{\beta-1}\right)^2}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^2}=\frac{1}{4}\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right]^2 \tag{69}$$

$$-\frac{\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}=\frac{\alpha}{2\beta}\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right] \tag{70}$$

$$-\frac{2\left(\frac{\beta-1}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-2}}{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)}=-\left(\frac{\beta-1}{\beta}\right)\left(\frac{\alpha}{x}\right)\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right] \tag{71}$$

Substitute equations (67), (69) and (71) into equation (66);

$$f''(x)=\left\{\begin{array}{l}-\left(\frac{\beta-1}{x^2}\right)+\frac{1}{2}\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right]^2\\-\left(\frac{\alpha(\beta-1)}{\beta x}\right)\left[\frac{\beta-1}{x}-\frac{f'(x)}{f(x)}\right]\end{array}\right\}f(x)+\frac{f'^2(x)}{f(x)} \quad (72)$$

The simplification of equation (72) yield the required ODE. Moreover two cases are considered.

Case I; When $\alpha=\beta=1$, equation (72) becomes;

$$f''(x)=\frac{3f'^2(x)}{2f(x)} \quad (73)$$

$$2f(x)f''(x)-3f'^2(x)=0 \quad (74)$$

See [11-24] for similar results.

B. Quantile Function

The Quantile function of the Log-logistic distribution is given by;

$$Q(p)=\alpha\left(\frac{p}{1-p}\right)^{\frac{1}{\beta}} \quad (75)$$

Differentiate equation (75);

$$Q'(p)=\left(\frac{\frac{1}{\beta}p^{\frac{1}{\beta}-1}}{p^{\frac{1}{\beta}}}+\frac{\frac{1}{\beta}(1-p)^{-\frac{1}{\beta}-1}}{(1-p)^{-\frac{1}{\beta}}}\right)Q(p) \quad (76)$$

$$Q'(p)=\left(\frac{1}{\beta p}+\frac{1}{\beta(1-p)}\right)Q(p)=\frac{Q(p)}{\beta}\left(\frac{1}{p}+\frac{1}{1-p}\right) \quad (77)$$

The first order ODE for the Quantile function of the Log-logistic distribution is given as;

$$\beta p(1-p)Q'(p)-Q(p)=0 \quad (78)$$

$$Q\left(\frac{1}{10}\right)=\alpha\left(\frac{1}{9}\right)^{\frac{1}{\beta}} \quad (79)$$

Differentiate equation (77) to obtain the second order ODE;

$$\beta Q''(p)=\left(\frac{1}{p}+\frac{1}{(1-p)}\right)Q'(p)+\left(-\frac{1}{p^2}+\frac{1}{(1-p)^2}\right)Q(p) \quad (80)$$

$$\beta Q''(p)=\frac{p(1-p)Q'(p)+(p^2-(1-p)^2)Q(p)}{p^2(1-p)^2} \quad (81)$$

The second order ODE for the Quantile function of the Log-logistic distribution is given as;

$$\begin{aligned}&\beta p^2(1-p)^2Q''(p)-p(1-p)Q'(p)\\&-(2p-1)Q(p)=0\end{aligned} \quad (82)$$

$$Q'\left(\frac{1}{10}\right)=\frac{100\alpha}{9\beta}\left(\frac{1}{9}\right)^{\frac{1}{\beta}} \quad (83)$$

Differentiate equation (80) to obtain the third order ODE;

$$\beta Q'''(p)=\left(\frac{1}{p}+\frac{1}{(1-p)}\right)Q''(p)+2\left(-\frac{1}{p^2}+\frac{1}{(1-p)^2}\right)Q'(p)+2\left(\frac{1}{p^3}+\frac{1}{(1-p)^3}\right)Q(p) \quad (84)$$

$$\beta Q'''(p)=\frac{Q''(p)}{p(1-p)}+\frac{2(p^2-(1-p)^2)Q'(p)}{p^2(1-p)^2}+\frac{2(p^3+(1-p)^3)Q(p)}{p^3(1-p)^3} \quad (85)$$

The second order ODE for the Quantile function of the Log-logistic distribution is given as;

$$\begin{aligned}&\beta p^3(1-p)^3Q'''(p)-p^2(1-p)^2Q''(p)\\&-2p(1-p)(2p-1)Q'(p)\\&-2(3p^2-3p+1)Q(p)=0\end{aligned} \quad (86)$$

$$Q''\left(\frac{1}{10}\right)=\frac{9200\alpha}{81\beta}\left(\frac{1}{9}\right)^{\frac{1}{\beta}} \quad (87)$$

See [11-24] for similar results.

C. Survival Function

The Survival function of the Log-logistic distribution is given by;

$$S(t)=\frac{1}{1+\left(\frac{t}{\alpha}\right)^{\beta}} \quad (88)$$

Differentiate equation (88);

$$S'(t)=-\frac{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{t}{\alpha}\right)^{\beta}\right)^2}=-f(t) \quad (89)$$

$$S'(t)+f(t)=0 \quad (90)$$

$$S(t)=\frac{\alpha^{\beta}}{\alpha^{\beta}+1} \quad (91)$$

The ODE can be obtained for any given parameters of the distribution. See [11-24] for similar results.

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E. Hazard Function

The Hazard function of the Log-logistic distribution is given by;

$$h(t)=\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t}{\alpha}\right)^{\beta}} \quad (92)$$

Differentiate equation (92);

$$h'(t)=\left\{\frac{\left(\frac{\beta-1}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-2}}{\left(\frac{t}{\alpha}\right)^{\beta-1}}-\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\left(1+\left(\frac{t}{\alpha}\right)^{\beta}\right)^{-2}}{\left(1+\left(\frac{t}{\alpha}\right)^{\beta}\right)^{-1}}\right\}h(t) \quad (93)$$

$$h'(t)=\left\{\frac{\beta-1}{t}-\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t}{\alpha}\right)^{\beta}}\right\}h(t) \quad (94)$$

$$h'(t)=\left(\frac{\beta-1}{t}-h(t)\right)h(t) \quad (95)$$

The first order ODE for the Hazard function of the Log-logistic distribution is given as;

$$th'(t)+th^2(t)-(\beta-1)h(t)=0 \quad (96)$$

$$h(1)=\frac{\beta}{\alpha^{\beta}+1} \quad (97)$$

See [11-24] for similar results.

IV. Concluding Remarks

Ordinary differential equations (ODEs) has been obtained for the probability density function (PDF), Quantile function (QF), survival function (SF) and hazard function (HF) of Logistic and log-logistic distributions. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the distributions determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [47-54]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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References

[1] J. Leydold and W. Hörmann, “ Generating generalized inverse Gaussian random variates by fast inversion,” *Comput. Stat. Data Analy.*, vol. 55, no. 1, pp. 213-217, 2011.

[2] G. Steinbrecher, G. and W.T. Shaw, “Quantile mechanics” *Euro. J. Appl. Math.*, vol. 19, no. 2, pp. 87-112, 2008.

[3] H.I. Okagbue, M.O. Adamu and T.A. Anake “Quantile Approximation of the Chi-square Distribution using the Quantile Mechanics,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 477-483.

[4] H.I. Okagbue, M.O. Adamu and T.A. Anake “Solutions of Chi-square Quantile Differential Equation,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 813-818.

[5] Y. Kabalci, “On the Nakagami-m Inverse Cumulative Distribution Function: Closed-Form Expression and Its Optimization by Backtracking Search Optimization Algorithm”, *Wireless Pers. Comm*. vol. 91, no. 1, pp. 1-8, 2016.

[6] W.P. Elderton, Frequency curves and correlation, Charles and Edwin Layton. London, 1906.

[7] N. Balakrishnan and C.D. Lai, Continuous bivariate distributions, 2nd edition, Springer New York, London, 2009.

[8] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2. 2nd edition, Wiley, 1995.

[9] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous univariate distributions, Wiley New York. ISBN: 0-471-58495-9, 1994.

[10] H. Rinne, Location scale distributions, linear estimation and probability plotting using MATLAB, 2010.

[11] H.I. Okagbue, P.E. Oguntunde, A.A. Opanuga, E.A. Owoloko “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Fréchet Distribution,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 186-191.

[12] H.I. Okagbue, P.E. Oguntunde, P.O. Ugwoke, A.A. Opanuga “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Generalized Exponential Distribution,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 192-197.

[13] H.I. Okagbue, A.A. Opanuga, E.A. Owoloko, M.O. Adamu “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Cauchy, Standard Cauchy and Log-Cauchy Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 198-204.

[14] H.I. Okagbue, S.A. Bishop, A.A. Opanuga, M.O. Adamu “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Burr XII and Pareto Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 399-404.

[15] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and A.A. Opanuga “Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gompertz and Gamma Gompertz Distributions,” In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 405-411.

[16] H.I. Okagbue, M.O. Adamu, A.A. Opanuga and J.G. Oghonyon “Classes of Ordinary Differential Equations Obtained for the

Probability Functions of 3-Parameter Weibull Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 539-545.

[17] H.I. Okagbue, A.A. Opanuga, E.A. Owoloko and M.O. Adamu "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Fréchet Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 546-551.

[18] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and S.A. Bishop "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Cauchy and Power Cauchy Distributions," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 552-558.

[19] H.I. Okagbue, P.E. Oguntunde, A.A. Opanuga and E.A. Owoloko "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponential and Truncated Exponential Distributions," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 858-864.

[20] H.I. Okagbue, O.O. Agboola, P.O. Ugwoke and A.A. Opanuga "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Pareto Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 865-870.

[21] H.I. Okagbue, O.O. Agboola, A.A. Opanuga and J.G. Oghonyon "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gumbel Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 871-875.

[22] H.I. Okagbue, O.A. Odetunmibi, A.A. Opanuga and P.E. Oguntunde "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Normal Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 876-882.

[23] H.I. Okagbue, M.O. Adamu, E.A. Owoloko and E.A. Suleiman "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Harris Extended Exponential Distribution," In Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 883-888.

[24] H.I. Okagbue, M.O. Adamu, T.A. Anake Ordinary Differential Equations of the Probability Functions of Weibull Distribution and their application in Ecology, *Int. J. Engine. Future Tech*., vol. 15, no. 4, pp. 57-78, 2018.

[25] N. Balakrishnan, *Handbook of the logistic distribution*. Marcel Dekker Inc. New York. ISBN 0-8247-8587-8., 2013.

[26] S.D. Dubey, "A new derivation of the logistic distribution", *Naval Res. Logistics Quart.*, vol. 16, no. 1, pp. 37-40, 1969.

[27] G.S. Mudholkar and E.O. George, "A remark on the shape of the logistic distribution", *Biometrika*, vol. 65, no. 3, pp. 667-668, 1978.

[28] S.R. Bowling, M.T. Khasawneh, S. Kaewkuekool and B.R. Cho, "A logistic approximation to the cumulative normal distribution", *J. Indust. Engine. Magt.*, vol. 2, no. 1, pp. 114-127, 2009.

[29] M.A. Stephens, "Tests of fit for the logistic distribution based on the empirical distribution function", *Biometrika*, vol. 66, no. 3, pp. 591-595, 1979.

[30] N. Balakrishnan and M.Y. Leung, "Order statistics from the type I generalized logistic distribution", *Comm. Stat. Simul. Comput.*, vol. 17, no. 1, pp. 25-50, 1988.

[31] S.S. Gupta and M. Gnanadesikan, "Estimation of the parameters of the logistic distribution", *Biometrika*, vol. 53, no. 3-4, pp. 565-570, 1966.

[32] S.S. Gupta, A.S. Qureishi and B.K. Shah, "Best linear unbiased estimators of the parameters of the logistic distribution using order statistics", *Technometrics*, vol. 9, no. 1, pp. 43-56, 1967.

[33] H.L. Harter and A.H. Moore, "Maximum-likelihood estimation, from censored samples of the parameters of a logistic distribution", *J. Amer. Stat. Assoc.*, vol. 62, no. 318, pp. 675-684, 1967.

[34] C. Antle, L. Klimko and W. Harkness, "Confidence intervals for the parameters of the logistic distribution", *Biometrika*, vol. 57, no. 2, pp. 397-402, 1970.

[35] R.E. Schafer and T.S. Sheffield, "Inferences on the parameters of the logistic distribution", *Biometrics*, vol. 29, no. 3, pp. 449-455, 1973.

[36] A. Asgharzadeh, "Point and interval estimation for a generalized logistic distribution under progressive type II censoring", *Comm. Stat. Theo. Meth.*, vol. 35, no. 9, pp. 1685-1702, 2006.

[37] M.M. Ali, N.N. Mikhail and M.S. Haq, "A class of bivariate distributions including the bivariate logistic", *J. Multivar. Analy.*, vol. 8, no. 3, pp. 405-412, 1978.

[38] E.O. George and M.O. Ojo, "On a generalization of the logistic distribution", *Ann. Inst. Stat. Math.*, vol. 32, no. 1, pp. 161-169, 1980.

[39] S.P. Satterthwaite and T.P. Hutchinson, "A generalisation of Gumbel's bivariate logistic distribution", *Metrika*, vol. 25, no. 1, pp. 163-170, 1978.

[40] M. Wang and K. Rennolls, "Tree diameter distribution modelling: introducing the logit logistic distribution", *Canadian J. Forest Res.*, vol. 35, no. 6, pp. 1305-1313, 2005.

[41] S. Nadarajah, "The skew logistic distribution", *AStA Adv. Stat. Analy.*, vol. 93, no. 2, pp. 187-203, 2009.

[42] H. Torabi and N.H. Montazeri, "The logistic-uniform distribution and its applications", *Comm. Stat. Simul. Comput.*, vol. 43, no. 10, pp. 2551-2569, 2014.

[43] M.H. Tahir, G.M. Cordeiro, A. Alzaatreh, M. Mansoor and M. Zubair, "The logistic-X family of distributions and its applications", *Comm. Stat. Theo. Meth.*, vol. 45, no. 24, pp. 7326-7349, 2016.

[44] A.K. Olapade, "The type I generalized half logistic distribution", *J. Iran. Stat. Soc.*, vol. 13, no. 1, pp. 69-82, 2014.

[45] G.M. Cordeiro, M. Alizadeh and P.R. Diniz Marinho, "The type I half-logistic family of distributions", *J. Stat. Comput. Simul.*, vol. 86, no. 4, pp. 707-728, 2016.

[46] R.R.L. Kantam, G. Srinivasa Rao and B. Sriram, "An economic reliability test plan: log-logistic distribution", *J. Appl. Stat.*, vol. 33, no. 3, pp. 291-296, 2006.

[47] S. Surendran, S. and K. Tota-Maharaj, "Log logistic distribution to model water demand data", *Proc. Engine.*, vol. 119, no. 1, pp. 798-802, 2015.

[48] A. A. Opanuga, H. I. Okagbue, E. A. Owoloko and O. O. Agboola, "Modified Adomian Decomposition Method for Thirteenth Order Boundary Value Problems", Gazi Univer. J. Sci., vol. 30, no. 4, pp. 454-461, 2017.

[49] A. A. Opanuga, E.A. Owoloko, H. I. Okagbue and O.O. Agboola, "Finite Difference Method and Laplace Transform for Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 65-69.

[50] A.A. Opanuga, H.I. Okagbue and O.O. Agboola "Application of Semi-Analytical Technique for Solving Thirteenth Order Boundary Value Problem," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, U.S.A., pp 145-148.

[51] A.A. Opanuga, E.A. Owoloko, H.I. Okagbue, "Comparison Homotopy Perturbation and Adomian Decomposition Techniques for Parabolic Equations," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 24-27.

[52] 7A.A. Opanuga, E.A. Owoloko, O.O. Agboola, H.I. Okagbue, "Application of Homotopy Perturbation and Modified Adomian Decomposition Methods for Higher Order Boundary Value Problems," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp. 130-134.

[53] A.A Opanuga, O.O. Agboola and H.I. Okagbue, "Approximate solution of multipoint boundary value problems", *J. Engine. Appl. Sci.*, vol. 10, no. 4, pp. 85-89, 2015.

[54] A.A. Opanuga, O.O. Agboola, H.I. Okagbue and J.G. Oghonyon, "Solution of differential equations by three semi-analytical techniques", *Int. J. Appl. Engine. Res.*, vol. 10, no. 18, pp. 39168-39174, 2015.

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